

# Neutrino Mass and Proton Lifetime in a Realistic SUSY $SO(10)$ Model

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## Abstract

This work presents a complete analysis of fermion fitting and proton decay in a supersymmetric  $SO(10)$  model previously suggested by Dutta, Mimura, and Mohapatra.

A key question in any grand unified theory is whether it satisfies the stringent experimental lower limits on the partial lifetimes of the proton. In more generic models, substantial fine-tuning is required among GUT-scale parameters to satisfy the limits. In the proposed model, the **10**,  **$\overline{126}$** , and **120** Yukawa couplings contributing to fermion masses have restricted textures intended to give favorable results for proton lifetime, while still giving rise to a realistic fermion sector, without the need for fine-tuning, even for large  $\tan\beta$ , and for either type-I or type-II dominance in the neutrino mass matrix.

In this thesis, I investigate the above hypothesis at a strict numerical level of scrutiny; I obtain a valid fit for the entire fermion sector for both types of seesaw dominance, including  $\theta_{13}$  in good agreement with the most recent data. For the case with type-II seesaw, I find that, using the Yukawa couplings fixed by the successful fermion sector fit, proton partial lifetime limits are readily satisfied for all but one of the pertinent decay modes for nearly arbitrary values of the triplet-Higgs mixing parameters, with the  $K^+\bar{\nu}$  mode requiring a minor  $\mathcal{O}(10^{-1})$  cancellation in order to satisfy its limit. I also find a maximum partial lifetime for that mode of  $\tau(K^+\bar{\nu}) \sim 10^{36}$  years. For the type-I seesaw case, I find that  $K^+\bar{\nu}$  decay mode is satisfied for any values of the triplet mixing parameters giving no major enhancement, and all other modes are easily satisfied for arbitrary mixing values; I also find a maximum partial lifetime for  $K^+\bar{\nu}$  of nearly  $10^{38}$  years, which is largely sub-dominant to gauge boson decay channels.

*For Erin, for my family, and for all the buds.*

*Booj.*

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# Chapter 1

## Introduction

The Standard Model of particle physics [1] is among the most fascinating of modern marvels, though it is an inconspicuous one. Its mathematical structure is capable of describing, with unparalleled precision, virtually every aspect of the statistical behavior of the elementary particles composing normal matter. With its last key aspects discovered by the early 1970's, the completed model emerged as the culmination of some forty years of effort to solve the many mysteries generated by the discoveries of quantum mechanics and relativity in the early 20th century.

Yet even as the final pieces were being put in place, physicists were already certain the model and its implications gave an incomplete version of the story of our universe: for as many questions as it answered with the utmost of elegance, the Standard Model (SM) left many mysteries unsolved and also gave rise to a few new ones. The model gives no indication as to why, in light of electroweak unification, there were still three separate forces in nature; in fact, it quite conspicuously gives no description of gravity, and further gives no explanation for dark matter or matter-antimatter asymmetry. Additionally it suggests that electric charge is quantized but provides no explanation for why it should be, nor does it relatedly give any reason for the values of hypercharge.

Furthermore, empirical evidence for other failures of the model were coming to light even before its completion. One important example of such evidence indicated a discrepancy in solar neutrino flux, which would ultimately come to be understood as a consequence of the oscillation of propagating neutrinos from one flavor to another [2]. It was already known at the time that such oscillations occur only among particles having mass, whereas the SM predicted neutrinos to be massless.

Thus, theorists began working to find an extension of the model that would solve



its problems without disrupting the beautiful predictions of its existing framework. One of the first notions to lead to some success was *Grand Unification* [3, 4], which nests the symmetry group of the SM in a higher dimensional group by expanding the potential (or superpotential) to include terms allowed by the higher dimensional symmetry; the new potential typically introduces heavy Higgs-like bosons and may include new multiplets of existing particles. Such a mathematical extension of the model is phenomenologically justified through the assumption that the “larger” symmetry of the Grand Unified theory would have been present at higher energies typical in the early universe, and that the SM symmetry would *emerge* at low energies through a *spontaneous breaking* of the larger symmetry. Grand Unified theory (GUT) provided understanding for some of the mysteries of the SM, and, when combined with the seesaw mechanism (see below) a few years later, it led to a nicely self-consistent and potentially testable explanation for neutrino masses and their apparent smallness. GUT framework again created some new questions of its own, and it also gave some curious predictions, such as the existence of proton decay [3].

Over the past few decades, and through the inclusion of Supersymmetry (SUSY) [5, 6], a few classes of GUT models, especially those based on the  $SO(10)$  symmetry group [7], have come to be realized as significantly more complete descriptions of our universe than the Standard Model. One of the more basic yet intriguing features of these models is the ability to naturally accommodate a right-handed neutrino, consequently allowing for a well-motivated implementation of the seesaw mechanism for neutrino mass [8, 9], a long-uncontested ansatz that dynamically explains the smallness of left-handed neutrino masses. The seesaw was originally implemented in the framework of SUSY  $SO(10)$  with **10**- and **126**-dimensional Higgs multiplets coupling to fermions [10, 11]; the vacuum expectation value (vev) of the **126** field plays the role of both breaking  $B-L$  and triggering the seesaw mechanism, thereby creating a deep mathematical connection between the smallness of neutrino masses and the other fermion masses. This seemingly limited yet elegant approach yielded a realistic neutrino sector, including an accurate prediction of the value of  $\theta_{13}$  [12, 13], long before experiments were measuring its value. In the SUSY context, it further provides a clear candidate for dark matter. This so-called “minimal”  $SO(10)$  model has been explored much more thoroughly over the years by many authors with the arrival of precision measurements [12–19], and it remains a viable predictor of the neutrino sector parameters.

Many of the remaining concerns associated with GUT models are on the verge of being addressed experimentally. Theorists and phenomenologists have made extensive

effort to carefully explore and catalogue in the vast number of feasible options available when constructing such a model, because each choice leads to a distinct set of favorable and unfavorable phenomenological features. It seems that within the next 10-20 years, this formidable tree of models will finally be pruned substantially as experiments close in on precise values for the phenomenological outputs whose predictions may distinguish one model from the next, including the remaining parameters of neutrino oscillation [20] and the lifetime of the proton [21].

Proton decay is arguably the most problematic feature common to nearly all GUT models. In all  $SU(5)$  and  $SO(10)$  models, heavy gauge boson exchanges give rise to effective higher-dimensional operators that allow for quark-lepton mixing and, consequently, nonzero probabilities for proton decay widths. Furthermore, in SUSY GUT models, although one sees an decrease in the decay widths following from gauge boson exchange, several additional decay modes are available, as each of the GUT-scale Higgs superfields contains colored Higgs triplets that allows for proton decay through exchange of Higgsino superpartners.

No one yet knows whether protons do in fact decay at all; if the answer turns out to be no, that will of course be the end of the line for GUT models without some new mechanism. So far, the lower limit on proton lifetime is known to be at least  $\sim 10^{33}$  years, and partial lifetimes for the various decay modes have been continually rising through the findings of experiments [22]. Thus, if any  $SO(10)$  model is to be trusted, its prediction for the proton lifetime must be at least so high a number. Most minimal  $SU(5)$  models have already been virtually ruled out by such limits.

There are ways in which the proton lifetime goal can be achieved within the framework of a given model, but doing so typically requires substantial fine-tuning, which occurs via rather extreme cancellations ( $\gtrsim \mathcal{O}(10^{-4})$ ) among the mixing parameters of the color-triplet Higgsinos exchanged in the decay. The values of those mixings cannot be reasonably recognized as more than arbitrary free parameters, so to expect multiple instances of very sensitive relationships among them requires putting much faith in either unknown dynamics or extremely good luck. Restricting the SUSY vev ratio  $v_u/v_d$ , conventionally parametrized as  $\tan \beta$ , to small values can provide some relief without cancellation for Higgsino-mediated decay channels, but such an assumption is still *ad hoc* and may ultimately be inconsistent with experimental findings; hence it is strongly preferable to construct a model which is tractable for any feasible  $\tan \beta$ .

If however the GUT Yukawas, which are  $3 \times 3$  matrices in generation space, have some key elements naturally small or zero, then extreme cancellations can be largely

avoided by eliminating most of the dominant contributions to proton decay width. A paper by Dutta, Mimura, and Mohapatra [23] proposed such a Yukawa texture for the  $SO(10)$  model that includes a **120** coupling in addition to the **10** and  $\overline{\mathbf{126}}$  Higgs contributions to fermion masses. The authors suggested that proton decay limits may be satisfied, especially for model with type-II seesaw dominance and sketched the relationships between key fermion fit parameters and proton partial lifetimes; however, the work gave mainly heuristic arguments and leading-order estimates to only tentatively support the hypothesis.

The work I present in this thesis revisits the above hypothesis and exposes it to robust testing by providing a careful and complete analysis of the characteristics of proton decay in the model. I grounded the analysis in conservative assumptions, including large  $\tan \beta$ , and performed a comprehensive numerical calculation relying on as few approximations as necessary. Furthermore, I extended the cursory work from ref. [23] for type-I seesaw to fully consider both the type-I and II seesaw dominance cases. The modes of proton decay that I checked for sufficiency are those known to be most problematic:  $p \rightarrow K^+ \bar{\nu}$ ,  $K^0 \ell^+$ ,  $\pi^+ \bar{\nu}$ , and  $\pi^0 \ell^+$ , where  $\ell = e, \mu$ .

The calculation consisted of two components: first I found a stable numerical fit to all fermion mass and mixing parameters, including the neutrino sector (where values are predictions of the model); then, using the Yukawa couplings fixed by the fermion fit as input, I searched the parameter space of heavy color triplet mixing parameters for areas that lead to adequately large partial lifetimes for the dominant modes of proton decay.

The results not only give satisfactory predictions for the neutrino sector based on corresponding charged sector fits, but also adequately predict sufficiently long-lived protons without relying on the usual large degree of tuning. I find that the ansatz is *completely successful* in satisfying the proton lifetime limits without any need for cancellation for the type-I seesaw scenario; a modest  $\mathcal{O}(10^{-1})$  cancellation is needed in the type-II case to satisfy the partial lifetime limit of the often-problematic  $p \rightarrow K^+ \bar{\nu}$  mode. These results for type-I versus type-II are contrary to the tentative expectations of the authors in [23]; the discrepancy is due mainly to the unexpected significance of the effect of rotation to mass basis on the results of the decay width calculations, combined with the numerical details of the rotation matrices arising from the charged sector mass and CKM fit.

The thesis is organized as follows. In chapter 2, I give an introduction to the Standard Model of particle physics and discuss its strengths and weaknesses. In chapter 3,

I give an introduction to supersymmetry and the Minimally Supersymmetric Standard Model (MSSM) and again discuss its strengths and weaknesses. In chapter 4, I give an overview of Grand Unified theories and their strengths and weaknesses and an introduction to  $SO(10)$  models; I also introduce the details of the model on which this work focuses, including the superpotential and the fermion mass matrices following from it, and the details of the Yukawa texture ansatz. In chapter 5, I expand further on the model specifics and examine general GUT proton-decay logistics in order to derive the needed partial decay widths. In chapter 6, I present the fermion sector results of the numerical fitting to the measured masses and mixings, and I present the results of the calculation of the important partial lifetimes of the proton. In chapter 7, I discuss the implications of the results and give my conclusions.

# Chapter 2

## The Standard Model

### 2.1 The Structure of the Standard Model

Strictly speaking, the Standard Model (SM) is a *spontaneously-broken non-Abelian gauge theory of quantum fields*. This extremely content-laden tagline can be parsed as follows.

A *quantum field* is a function over some space or spacetime that assigns an algebraic operator, rather than a numerical value, to each point in the space. Such an operator typically acts on elements of a separate internal vector space; that action creates (or destroys) discrete excited states of the underlying field called *quanta*. The actions of multiple operators are not generally commutative.

In relativistic quantum field theory, *elementary particles* are realized as excitations in *Fock space*, which is a generalization of the (non-relativistic) quantum-mechanical Hilbert space that allows for the accommodation of multi-particle states in which the number of particles is not fixed. The “value” of a typical (scalar) quantum field  $\phi$  at a spacetime point  $x$  goes like  $e^{ip \cdot x} \hat{a}^\dagger |0\rangle$  or  $e^{-ip \cdot x} \hat{a} |0\rangle$ , where  $\hat{a}^\dagger$  is the raising operator (like that of a harmonic oscillator) whose action on the Fock space *ground state*  $|0\rangle$  (“the vacuum”) creates a single quantum of the field. The new state  $\hat{a}^\dagger |0\rangle$ , explicitly notated as “ $|1\rangle$ ” or, more commonly, “ $|p\rangle$ ”, is identified with a plane wave carrying momentum  $p$ , “pinned” to spacetime at the point  $x$ , and it can be further associated with a *representation* of the *Lorentz group*,  $SO(1,3)$ , which I will describe in detail shortly. The lowering operator  $\hat{a}$  acting on  $|p\rangle$  destroys a single field quantum, while  $\hat{a}^\dagger \hat{a}^\dagger |0\rangle$  creates two quanta, corresponding to a two-particle state  $|p_1 p_2\rangle$ , and so on. Note though that states of more than one identical particles are forbidden for fermionic

fields due to the Pauli exclusion principle. As with any lowering operator,  $\hat{a}|0\rangle = 0$ .

Both “non-Abelian” and “gauge” theories of quantum fields are types of *group theories*. A *group* is a set of elements, together with an associative operation, that

- is *closed* under the action of the operation on any two elements
- contains a unique *identity* element
- contains a unique *inverse* for every element.

The set of elements of a group can be finite and discrete, countably infinite, or a continuous spectrum. A simple example of a group is the integers with the addition operation  $\{\mathbb{Z}, +\}$ , where zero is the identity element and negative integers are the inverse elements of positive integers (and vice versa).

If the elements of a continuous group also form a topological *manifold* (i.e., if the space is “smooth”, or continuous and differentiable throughout), then the group is known as a *Lie group*.

A *non-Abelian* group is a group (finite or continuous) for which the group operation is non-commutative on two elements; i.e., for elements  $a, b$  of a group  $\{G, \cdot\}$ ,  $a \cdot b \neq b \cdot a$ .

Before I can give proper discussions of the remaining terms in this “mathematical name” for the Standard Model, I will need to introduce quite a bit of additional terminology.

A group *representation* is a map from a group  $G$  to a set of linear transformations on a vector space  $V$ . More explicitly, the map  $\pi$  is a homomorphism

$$\pi : G \longrightarrow GL(V)$$

with the property

$$\pi(g \cdot h) = \pi(g) \circ \pi(h) \quad \text{for } g, h \in G;$$

$GL(V)$  is the general linear group (a group in its own right) consisting of all  $N \times N$  matrices acting on an  $N$ -dimensional vector space  $V$ ; thus the representation of a group  $\pi(G)$  is always some subgroup of  $GL(V)$ . If the homomorphism  $\pi$  is one-to-one, (*injective*), then the map is an isomorphism:  $G \cong \pi(G)$ , and the representation is said to be *faithful*.

A representation is conventionally named simply with a bold numeral indicating its dimension, as in, for example, the “**2**” or the “**3**” representation of  $SU(2)$ . In a

mild abuse of terminology, physicists are quite prone to referring to a vector  $v \in V$ , on which the elements of a group representation act, as a “representation” of the group as well; in fact, I will often do so in this work.

When a mathematical system is left unchanged by the simultaneous action of a group on each of the components of the system, the group is called a *symmetry* of the system, and the system is said to be *invariant* under the group action.

To qualify the above concepts in the pertinent context, let me point out that the Lagrangian of the Standard Model is invariant under the action of the continuous group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \times (\mathbb{R}^{1,3} \rtimes SO(1,3)),$$

where

- $SO(N)$  is the non-Abelian group of *orthogonal* (i.e., length-preserving) rotations in  $N$ -dimensions, with elements  $O$  such that  $O^T O = \mathbb{I} \forall O \in SO(N)$ ; it is naturally equipped with the *fundamental* or *standard*<sup>1</sup> representation of  $N \times N$  matrices satisfying the above property and with determinant 1, which act on vectors in the space  $\mathbb{R}^N$ .
- $SU(N)$  is the analogous group of complex *unitary* rotations with elements  $U$  such that  $U^\dagger U = \mathbb{I} \forall U \in SU(N)$ , and with fundamental representation acting on elements of the *complex* space  $\mathbb{C}^N$ .
- $U(1)$  is the Abelian group of rotations by a complex phase  $e^{i\theta}$  for some real number  $\theta$ , which acts on single elements of  $\mathbb{C}$ , i.e., complex numbers.
- the direct products “ $\times$ ” indicate that, although the individual groups are generally non-Abelian, the actions of the groups commute with one another.
- $\mathbb{R}^{1,3} \rtimes SO(1,3)$  is the *Poincaré* group, the “spacetime part” of the SM symmetry. Poincaré invariance is what makes the SM consistent with the principles of special relativity.  $\mathbb{R}^{1,3}$  gives the translational symmetry of any SM process (i.e., the physics is the same whether some interaction happens at point  $x$  or point  $y$ ), and  $SO(1,3)$ , the *Lorentz* group, contains ordinary rotations in 3D space plus boosts (time-space mixing rotations). The presence of the *semi-direct product*, “ $\rtimes$ ”, is due to the fact that the product of an  $SO(1,3)$  transformation and an  $\mathbb{R}^{1,3}$  translation is another translation in a different reference frame; hence, for a general

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<sup>1</sup>“Standard representation” is the conventional term among mathematicians.

spacetime translation  $U \sim e^{ip \cdot x}$  and a general spacetime rotation  $\Lambda \in SO(1, 3)$ , the commutator  $U \cdot \Lambda - \Lambda \cdot U \sim U'$  is nonzero (i.e., they do not commute). The *signature* “1,3” carries the distinction between timelike and spacelike directions; the two have opposite-sign contributions to the metric  $\eta_{\mu\nu}$  used to calculate inner products between elements of the Poincaré group, which creates the potential for *null*, or “light-like” propagation, for which the invariant spacetime *interval*  $ds^2 \equiv \eta_{\mu\nu} x^\mu x^\nu = dt^2 - dx^2 = 0$ .<sup>2</sup>

Note that  $SO(N)$ ,  $SU(N)$ , and  $U(1)$  are all Lie groups.

A *Lie algebra*  $\mathfrak{g}$  is related to the Lie group  $G$  by the following rule: for all  $N \times N$  matrices  $X \in \mathfrak{g}$  and  $\theta \in \mathbb{R}$ ,  $U = e^{i\theta X} \in G$ . Note that the factor of  $i$  is a practical convention used by physicists. The real parameter  $\theta$  sets the magnitude for the group transformation (extraction of this factor from  $X$  is not necessary, but it is convenient and will be easier to generalize later); in the cases of orthogonal or unitary transformations, it can be interpreted as a rotation angle. If  $\theta \ll 1$ , then  $U$  can be simplified using the infinitesimal form of the exponential  $U \approx 1 + i\theta X$ .

The *generators* of a Lie algebra  $t^a$  are the basis elements through which all  $X \in \mathfrak{g}$  can be constructed; i.e.,  $X = \sum \alpha^a t^a \ \forall X \in \mathfrak{g}$ , with  $\alpha^a \in \mathbb{R}$ . By the relationship given in the previous paragraph, any element of the group can be written as  $U = e^{i\alpha^a t^a}$ , where the rotation angle has been absorbed into the constants  $\alpha$ . This is a general form for the elements of  $SU(N)$  in the SM; their action on fermion fields is  $\psi \rightarrow U\psi$ .

The  $N(N - 1)/2$  generators of the Lie algebra  $\mathfrak{so}(N)$  are antisymmetric, and the  $N^2 - 1$  generators of  $\mathfrak{su}(N)$  are Hermitian. The closure of  $G$  is guaranteed if the generators of  $\mathfrak{g}$  satisfy the commutator relationship

$$[t^a, t^b] \equiv t^a t^b - t^b t^a = i f^{ab}{}_c t^c,$$

where  $f^{ab}{}_c$  are called the *structure constants* of the algebra. The structure constants are simply numbers that determine the exactly how one generator is constructed from the others. It is naturally the case that many of the structure constants for a particular Lie algebra are zero.

Here I can finally return to the defining the terms appearing in the opening sentence. A *gauge symmetry* is an invariance under *local* group transformations, as opposed to *global* transformations. In a global transformation, the rotation parameters  $\alpha^a$  are

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<sup>2</sup>I will use the “mostly minus” signature, with spacelike elements of the metric negative, i.e.  $\eta \equiv \text{diag}(1, -1, -1, -1)$ .



constant real numbers, as described above. In a local transformation, the parameters are instead functions of spacetime,  $\alpha^a = \alpha^a(x)$ , which is actually a stronger condition (*i.e.*, local symmetry implies global symmetry).

This promotion of transformations has surprising effects on the nature of a theory. Before trying to understand gauge symmetry in a quantum field theory, I will consider a simple example from classical electromagnetism. One may recall that an electromagnetic wave has only two degrees of freedom, namely the polarizations of  $\mathbf{E}$  and  $\mathbf{B}$ ; yet, the four-vector potential  $A_\mu$ , whose spacetime derivatives give rise to those fields, seemingly comes equipped with four degrees of freedom. Thus it seems the potential has some intrinsic redundancy; in fact, that redundancy follows directly from the ambiguity in its definition:

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha, \quad (2.1)$$

where  $\alpha(x)$  is some scalar function (the degeneracy of this notation with that of the gauge transformation parameters is intentional). Furthermore, the Lagrangian for  $A_\mu$ , from which Maxwell's equations follow,  $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ , is invariant under the redefinition (2.1). This is a simple example of a gauge symmetry.

As it turns out,  $A_\mu$  is a representation of the Lorentz group, and precisely that which one would promote to an operator if looking to quantize electromagnetism. If one naively attempts to do so by, for instance, following procedure analogous to that for a scalar field, serious difficulties arise presently. Given the equation of motion for the classical photon-to-be,

$$\partial_\mu F^{\mu\nu} = \partial_\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = J^\nu,$$

or, after Fourier transform,

$$(-p^2 g_{\mu\nu} + p_\mu p_\nu) A_\mu = J^\nu,$$

one finds that the naive choice for the corresponding propagator is ill-defined. However, one can utilize the ambiguity in (2.1) to resolve the issue by adding a term that depends on the “choice of gauge”, *i.e.* the form of  $\alpha(x)$  (or, traditionally, an analogous function). In the end one sees that the lack of an “ordinary” propagator is a consequence of neglecting the redundancy of the extraneous degrees of freedom. Therefore, any quantized theory of electromagnetism will necessarily also be a gauge theory.

One important consequence of this generalization is that terms in the Lagrangian containing derivatives of matter fields are no longer invariant under group transformations. For the Abelian group  $U(1)_{\text{em}}$  of proper quantum electrodynamics (QED), a term involving matter fields such as  $\bar{\psi}\psi$  (more on this form later...) is unchanged by the transformation  $\psi \rightarrow e^{i\alpha}\psi$  even after the “gauging” of the symmetry,  $\alpha \rightarrow \alpha(x)$ , because the transformation factors enter as conjugates and simply cancel; however, the derivative transformation picks up an extra term:

$$\partial_\mu\psi \rightarrow e^{i\alpha(x)}\partial_\mu\psi + i\partial_\mu\alpha e^{i\alpha(x)}\psi.$$

In order to restore invariance to derivative terms in the Lagrangian, one must introduce the *gauge covariant derivative*  $D_\mu \equiv \partial_\mu + iA_\mu$ . Using this form in place of the normal derivative, as well as the transformations for both  $A_\mu$  and  $\psi$ , one finds that  $D_\mu\psi \rightarrow e^{i\alpha(x)}D_\mu\psi$ , as desired. The details of the Lagrangian in light of this formulation will be discussed in more detail later. The generalization of this process to non-Abelian groups is relatively straightforward.

As the final topic from my opening remark, a *spontaneously broken* symmetry is a symmetry of the Lagrangian that is not respected by the ground state of the theory. In the case of the SM, the  $SU(2)_L \times U(1)_Y$  electroweak symmetry is not a symmetry of the vacuum. The symmetry is “broken” (really more like obscured) specifically by the Higgs field via the *Higgs mechanism* at the electroweak scale  $\sim 100$  GeV. I will discuss the Higgs mechanism and the implications of this symmetry breaking in more detail shortly.

At this point, all of the terminology I used at the start of the chapter to name the mathematical structure of the SM has been introduced. Before discussing the Lagrangian and the interactions at the heart of the model, I will discuss the details of representations of the SM fields.

### 2.1.1 The Representations of Standard Model Fields

The SM includes the following quantum fields:

- three copies of four fermionic fields:  $3 \times 2$  *quark* fields,  $\{u, c, t\}$ , and  $\{d, s, b\}$ , and  $3 \times 2$  *lepton* fields,  $\{\nu_e, \nu_\mu, \nu_\tau\}$ , and  $\{e, \mu, \tau\}$ ; the “copies”, known as *generations*, differ only in mass and have the same quantum numbers otherwise;

- four force-carrying bosonic fields: the photon,  $A_\mu$  (often notated as “ $\gamma$ ”), the gluons,  $G_\mu^a$  (often notated as “ $g$ ”), and the  $W_\mu^\pm$  and  $Z_\mu$  weak bosons;
- one Higgs boson field,  $\phi$ .

The force-carrying bosons named here are the physical particles, of definite mass, which differ from the massless fields found in the model prior to spontaneous symmetry breaking. Those fields will be discussed shortly, and their relationships to the above particles will be made clear when I discuss symmetry breaking in more detail.

Each field above is associated to a particular representation of the SM gauge group (gauge bosons) or the vector spaces on which it acts (matter fermions and Higgs). Differences in representation are what give the fields unique properties, which lead to our observation of several unique types of elementary particles. Below I will discuss the representations for each field.

## Spacetime Representations

The different classes of fields listed above experience spacetime transformations as different representations of the Poincaré group, which, in a sense, gives rise to the simplest definition of *elementary particle*: a state whose degrees of freedom mix only with each other, as elements of a single representation, under the action of the Poincaré group, [24]. Furthermore, the nature of translation is generic to all of the fields, so it is specifically the Lorentz representation of a particle that determines the nature of the interactions it may have, and even the nature of its free propagation through empty space.

**Lorentz Scalars.** The most basic and uninteresting Lorentz representation is the *trivial* representation; fields in this representation are invariant under group transformations and are consequently scalars in the formalism of the group.<sup>3</sup> The Higgs boson is the only Lorentz scalar field in the SM.

**Lorentz Vectors.** The force-carrier gauge bosons of the SM are Lorentz *four-vectors*, i.e., 3+1-dimensional elements of the fundamental representation; for the Lorentz group, this implies transformation via the same  $4 \times 4$  boost or rotation matrices as  $x^\mu$ ,  $p^\mu$ , etc. one sees in basic index-notated special relativity:  $A'_\nu = \Lambda_\nu^\mu A_\mu$ .

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<sup>3</sup>Note the concept of a trivial representation is general to all groups and is not a special feature of the Lorentz group.

**Spinors.** The matter fermions of the SM are Lorentz or Dirac *spinors*. A spinor representation is also realized as matrices acting on multiplets in a vector space, but it is a different vector space, of generally different dimension, from that of the fundamental representation. The relationship between the two spaces is an interesting one. The group  $Spin(N)$ , whose elements act on the spinors, is a *double cover* of the orthogonal group  $SO(N)$ , meaning there are two “copies” of the  $SO(N)$  manifold in that of  $Spin(N)$ , and there is a 2-to-1 map from the latter onto the former. As a result, for any rotation of a vector in the space of the  $SO(N)$  fundamental, there are two topologically *distinct* continuous paths, from the same initial state to the same final state, through which the spinor can be rotated. Another important result of this relationship is that an ordinary spatial rotation of a spinor through  $2\pi$  results in the *negative* of the original state; a second  $2\pi$  rotation is required to return the spinor to its original orientation.

For the Lorentz group, the double covering group is  $Spin(1, 3) \cong SL(2, \mathbb{C})$ , which is the *special linear* group over complex numbers, whose elements are  $2 \times 2$  matrices with complex entries and determinant 1. The action of  $SL(2, \mathbb{C})$  is on two-component *Weyl* or *chiral* spinors  $\psi_{L,R}$ ; the Dirac spinor more commonly associated with the Lorentz group is actually a *bispinor*, spinor  $\oplus$  spinor; this reducibility is manifest in the *Weyl basis* for the gamma matrices, where the bispinor corresponding to a SM fermion is the direct sum  $\psi = \psi_L \oplus \psi_R$ ; many interactions of bispinors, including those in QED, decouple into left and right parts in that basis. Four-component Dirac “spinors” are related to Weyl bispinors by a change of basis.

Interaction of spinors with a Lorentz vector is realized through the *Dirac algebra*, which consists of  $4 \times 4$  matrices  $\gamma^\mu$  that form an anti-commuting *Clifford algebra*, meaning they satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}_4,$$

where  $\mathbb{I}_4$  is the identity in the spin space. Note that each matrix carries a Lorentz spacetime index, which can have values  $\mu = 0, 1, 2, 3$  as one would expect; yet, the  $\gamma$ -matrices are better thought of as a basis for representing four-vectors as group elements in the spin space (*i.e.*, matrix operators that act on spinors), rather than as forming a spacetime four-vector themselves, especially as they transform differently (and passively) under the Lorentz group.

In analogy with non-relativistic angular momentum, the six objects

$$S^{\mu\nu} \equiv \frac{1}{2}\gamma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu],$$

are the generators of angular momentum and boosts in the spin space; accordingly,  $S^{\mu\nu}$ , rather than the  $\gamma$ -matrices themselves, satisfy the Lie algebra  $\mathfrak{so}(1,3)$ , and hence represent the group  $Spin(1,3)$ . The Lorentz transformation of a Dirac spinor is given in terms of these generators:

$$\psi \rightarrow \Lambda_{\frac{1}{2}} \psi = \exp \left( -\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \psi,$$

where  $\omega_{\mu\nu}$  is an anti-symmetric tensor of constant infinitesimal rotation parameters. This Lorentz transformation for spinors is related to the vector transformation  $\Lambda^\mu{}_\nu$  through the gamma matrices:

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu.$$

Before I move on, note that the Lorentz invariant contraction of spinors is

$$\bar{\psi}\psi \equiv \psi^\dagger \gamma^0 \psi = \psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R,$$

rather than the naive choice of  $\psi^\dagger \psi$ . It will generally be the case that Lorentz tensors constructed from spinors will involve some product of gamma matrices sandwiched between  $\bar{\psi}$  and  $\psi$ : the vector  $\bar{\psi} \gamma^\mu \psi$ , which couples to ordinary Lorentz vectors, the pseudo-vector  $\bar{\psi} \gamma^\mu \gamma^5 \psi$ , the two-tensor  $\bar{\psi} \gamma^{\mu\nu} \psi$ , etc.

## Representations of the Internal Gauge Group

All three components of the *internal* symmetry group of the SM are gauged groups. Fermionic matter fields transform under the action of the fundamental representations of those groups; *i.e.*, the fields are components of an  $N$ -dimensional multiplet on which a group  $SU(N)$  acts in the form of an  $N \times N$  matrix.

In particular, fermions with left-handed chirality are known to pair off into doublets,

$$q \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \ell \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

which can be rotated by  $SU(2)$  group elements; gauge covariance of the group leads to interactions between the left-handed fermion multiplets above and the  $W$  bosons, giving rise to the weak force, although the details are complicated a bit by electroweak symmetry breaking (EWSB). The transformations are associated with left-handed fermions

having non-trivial *weak isospin* charge,  $\mathbf{T}$ . Right-handed fermions,  $u_R$ ,  $d_R$ , and  $e_R$ , have  $\mathbf{T} = 0$ , and so each exists only in the trivial representation of  $SU(2)$ . In analogy with ordinary spin, the components of each doublet have eigenvalues  $T^3 = \pm 1/2$ .

Similarly, quarks possess an additional degree of freedom known as *color* and consequently form triplets,

$$u = \begin{pmatrix} u_r \\ u_g \\ u_b \end{pmatrix} \quad d = \begin{pmatrix} d_r \\ d_g \\ d_b \end{pmatrix},$$

which can be rotated by  $SU(3)$  group elements; gauge covariance of the group gives rise to the strong force through interactions between the quark multiplets above and the gluons. Leptons do not carry color charge and so are found in the trivial representation of this group. Interestingly enough, every known physical state involving quarks which has been empirically verified is *color neutral*, or “white”; individual quarks do not freely propagate at low energies. This property of quarks, known as *confinement*, is perhaps not yet fully understood, but is due in part to the fact that the strength of the coupling constant  $g_s$  for color interactions increases as energy decreases.

Finally, all fermionic SM fields individually have nonzero *weak hypercharge*,  $Y_w$ , which is associated with rotations by group elements of the  $U(1)_Y$  symmetry; gauge covariance of the group ultimately gives rise to the electromagnetic force through interactions between fermions and photons, although, again, the details are complicated by EWSB. The transformations act on individual fields rather than multiplets, meaning the group elements are simply complex numbers of unit magnitude.

The corresponding antiparticle fields of the SM fermions, which are the charge conjugates of the particle fields, are found in analogous *conjugate* representations, named “ $\bar{\mathbf{2}}$ ”, “ $\bar{\mathbf{3}}$ ”, etc.; the antiparticle partners themselves are named by one of a few conventions. One often sees the notation  $\psi^c \equiv C\bar{\psi}^T = C\gamma^0\psi^*$  to indicate antiparticle fields, where the  $C$  is a unitary matrix with  $C^T = -C$ ; by this construction, the antiparticle  $\psi^c$  has the same chirality as its partner  $\psi$ . Once I move on from discussing the SM, I will normally use this notation. Note though that if I want to give the antiparticle partners of the  $SU(2)_L$  doublets above, I would write something like

$$q_R^\dagger \equiv \begin{pmatrix} u_R^\dagger & d_R^\dagger \end{pmatrix}, \quad \ell_R^\dagger \equiv \begin{pmatrix} \nu_R^\dagger & e_R^\dagger \end{pmatrix},$$

to make manifest that only antiparticles with right-handed chirality will form  $SU(2)_L$  doublets that interact via the weak force.

Force-carrier gauge bosons experience (and, in a way, exhibit) the action of the internal symmetry groups of the SM as elements of the *adjoint* representations of the groups; the adjoint representation is that which is exhibited by the generators of the Lie algebra themselves; the group action on the generators is  $t^a \rightarrow g t^a g^{-1}$  for some  $g \in \text{group } G$ ; more specifically for our purposes,  $t^a \rightarrow U t^a U^\dagger$  for  $U \in SU(N)$ . The boson fields  $A_\mu^a(x)$  associated with a particular symmetry group will be in one-to-one correspondence with the generators of the symmetry. For a gauge symmetry, the transformation of the bosons mimics that of the generators, but with an important extension:  $A^a \rightarrow U A^a U^\dagger + dU U^\dagger$ ; taking  $U = e^{-i\alpha^a t^a}$  as before, and for infinitesimal transformations  $\alpha(x) \ll 1$ , this corresponds to  $A_\mu^a \rightarrow A_\mu^a + \partial_\mu \alpha^a - f_{bc}^a \alpha^b A_\mu^c$ , which is the generalization of eq. (2.1) for the abelian gauge field  $A_\mu$  discussed earlier. The generalized gauge covariant derivative for a non-Abelian group utilizes the above properties to give the mapping of the boson field into the vector space of the group:  $D_\mu = \partial_\mu - i g t^a A_\mu^a$ , where  $g$  is the coupling constant of the interaction with other fields; interactions with matter fields arise through this *minimal coupling* of the gauge field to the derivative.

The vector bosons associated with the unbroken symmetry of the SM are the single field  $B_\mu$  for the Abelian group  $U(1)_Y$ , the three fields  $W_\mu^a$  for  $SU(2)_L$ , and the eight *gluons*  $G_\mu^{a'}$  for  $SU(3)_c$ .

The scalar Higgs field  $\phi$  is an  $SU(2)_L$  doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad (2.2)$$

with hypercharge  $Y_w = 1/2$ . Each component field is complex, so  $\phi$  generally has 4 degrees of freedom. The non-trivial  $SU(2)_L$  representation enables electroweak symmetry breaking when the field acquires a vacuum expectation value, which I will discuss in more detail shortly. Additionally, the field belongs to the trivial representation of  $SU(3)_c$ .

A summary of the charges of all the SM fields under each symmetry group is given in Table 2.1.

	$SU(3)$ rep	$SU(2)$ rep	$Y_w$
$q_L^i$	<b>3</b>	<b>2</b>	1/6
$u_R^i$	<b>3</b>	<b>1</b>	2/3
$d_R^i$	<b>3</b>	<b>1</b>	-1/3
$\ell_L^i$	<b>1</b>	<b>2</b>	-1/2
$e_R^i$	<b>1</b>	<b>1</b>	-1
$B_\mu$	<b>1</b>	<b>1</b>	0
$W_\mu^a$	<b>1</b>	<b>3</b> (adj)	0
$G_\mu^{a'}$	<b>8</b> (adj)	<b>1</b>	0
$\phi$	<b>1</b>	<b>2</b>	1/2

Table 2.1: Representations and charges of SM fields under the internal gauge symmetries of the model.

### 2.1.2 Standard Model Interactions and Lagrangian

In accordance with classical Lagrangian theory, the SM Lagrangian should incorporate all of the allowed dynamics of its particles in terms of only the fields and their spacetime derivatives. A properly formed Lagrangian density  $\mathcal{L}$  should be such that the action  $\mathcal{S} \equiv \int d^4x \mathcal{L}$  is invariant under a general transformation of either the Poincaré group or the internal SM gauge group (at least up to some total derivative), which implies that each term in  $\mathcal{L}$  should be written in such a way that all of its components are contracted to result in a scalar under general transformations. Also, it follows from  $\mathcal{S}$  (and  $\hbar = 1$ ) that  $\mathcal{L}$  must have dimensions of energy<sup>4</sup>.

In classical field theory, kinetic terms are  $\sim (d\Phi)^2$ . For a scalar quantum field  $\phi$  (of dimension  $[\phi] = 1$ ), the analogy is exact:  $\mathcal{L}_{\text{kin}} = (\partial_\mu \phi)^2$ , where there is an implied sum over  $\mu$  (note  $[\partial_\mu] = [p^\mu] = 1$  also, so that  $[\mathcal{L}_{\text{kin}}] = 4$  as desired). The generalization for a complex field (like the Higgs) is  $\partial^\mu \phi^* \partial_\mu \phi$ . I mentioned the kinetic Lagrangian for the Abelian  $A_\mu$  field in the earlier discussion on gauge symmetry; the generalization to non-Abelian bosons follows from  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{bc}^a A_\mu^b A_\nu^c$ . Note, one can see from this expression that non-Abelian bosons interact among themselves, *i.e.*, they carry charge under the force they mediate, which is not the case for electrically-neutral photons. The resulting kinetic terms for the SM Lagrangian are

$$\mathcal{L}_{SM} \ni -\frac{1}{4}G_{a'}^{\mu\nu}G_{\mu\nu}^{a'} - \frac{1}{4}W_a^{\mu\nu}W_{\mu\nu}^a - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}, \quad (2.3)$$

where  $B_{\mu\nu}$  is analogous to the Abelian electromagnetic field strength tensor  $F_{\mu\nu}$ .



The kinetic term for fermion fields is a bit more tricky. For one, Dirac spinors have dimension  $[\psi] = 3/2$ , so the operator in question will need to contain only a single derivative; furthermore, that derivative will still need to be contracted with another vector-like object. The solution, courtesy of Dirac, turns out to be  $i\bar{\psi}\gamma^\mu\partial_\mu\psi$ . Note one often sees *Feynman slash notation*  $\not{p} = \gamma^\mu p_\mu$  for contraction of four-vectors with the gamma matrices.

The interaction terms for scalars or spinors with the gauge bosons follow straightforwardly from replacing the derivatives above with the corresponding gauge covariant derivatives. The components of the Lagrangian consistent with the representations described in the previous section are

$$\begin{aligned}
\mathcal{L}_{SM} \ni & \bar{q}_L^i \gamma^\mu \left( i\partial_\mu + g_s \lambda^{a'} G_\mu^{a'} + g T^a W_\mu^a + \frac{1}{6} g' B_\mu \right) q_L^i \\
& + \bar{u}_R^i \gamma^\mu \left( i\partial_\mu + g_s \lambda^{a'} G_\mu^{a'} + \frac{2}{3} g' B_\mu \right) u_R^i \\
& + \bar{d}_R^i \gamma^\mu \left( i\partial_\mu + g_s \lambda^{a'} G_\mu^{a'} - \frac{1}{3} g' B_\mu \right) d_R^i \\
& + \bar{\ell}_L^i \gamma^\mu \left( i\partial_\mu + g T^a W_\mu^a - \frac{1}{2} g' B_\mu \right) \ell_L^i \\
& + \bar{e}_R^i \gamma^\mu (i\partial_\mu - g' B_\mu) e_R^i \\
& + \phi^\dagger \left( \partial^\mu + ig T^a W_\mu^a + \frac{i}{2} g' B^\mu \right) \left( \partial_\mu - ig T^b W_\mu^b - \frac{i}{2} g' B_\mu \right) \phi,
\end{aligned} \tag{2.4}$$

where the generators  $T^a \equiv \sigma^a/2$ , with  $a' = 1, 2, 3$ , are half the Pauli matrices;  $\lambda^{a'}$ , with  $a = 1, \dots, 8$ , are the analogous generators of  $SU(3)$ ; and  $i = 1, 2, 3$  are the generation indices, for which all of the above interactions are diagonal (in the unbroken, massless case). In this context the spinor fields  $f_{L,R}$  with  $f = u, d, e, \nu$  are four-component Dirac spinors, rather than two-component Weyl spinors, but with the left- or right-handed components set to zero, which can be done using the *chiral projection operators*  $P_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)$  such that  $f_{L,R} = P_{L,R}f$ . Note the quark-lepton asymmetry due to the absence of the right-handed neutrino field. The implicit transpose in  $\phi^\dagger$  is with respect to its  $SU(2)$  components, and the adjacent derivative acts on it to the left. Also note that the indices for the internal spaces of  $SU(2)$  and  $SU(3)$  have been suppressed for clarity; for example, the fully notated version of the quark doublet term above would be

$$\mathcal{L} \ni \bar{q}_L^{i\alpha\rho} \gamma^\mu \left( \delta_{\alpha\beta} \delta_{\rho\sigma} (i\partial_\mu + \frac{1}{6} g' B_\mu) + g_s \delta_{\alpha\beta} \lambda_{\rho\sigma}^{a'} G_\mu^{a'} + g \delta_{\rho\sigma} T_{\alpha\beta}^a W_\mu^a \right) q_L^{i\beta\sigma},$$

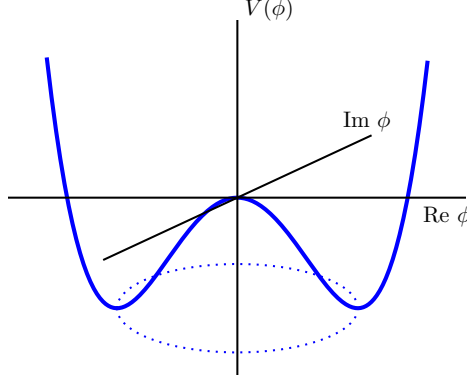


Figure 2.1: The classical potential for the Higgs field as a function of  $\phi$ .

where  $\alpha = 1, 2$  are the internal  $SU(2)$  indices, and  $\rho = 1, 2, 3$  are those of  $SU(3)$ .

The Higgs field  $\phi$  also interacts with the matter fields through the *Yukawa* terms, and has self-interactions allowed by the freedom of the Lorentz scalar representation as well:

$$\begin{aligned}
\mathcal{L}_{SM} \ni & -y_u^{ij} \epsilon_{\alpha\beta} \bar{q}_{Li}^{\alpha} \phi^{*\beta} u_{Rj} - y_d^{ij} \bar{q}_{Li}^{\alpha} \phi^{\alpha} d_{Rj} - y_e^{ij} \bar{\ell}_{Li}^{\alpha} \phi^{\alpha} e_{Rj} \\
& + \text{Hermitian conjugates} \\
& + \mu^2 \phi^{\dagger} \phi - \lambda (\phi^{\dagger} \phi)^2,
\end{aligned} \tag{2.5}$$

where I've included the  $SU(2)$  indices in the Yukawa terms due to their non-triviality. Note that  $\epsilon_{\alpha\beta} \phi^{*\beta}$  (with  $\epsilon_{12} = 1$ ) transforms identically to  $\phi$  under  $SU(2)$  but has the opposite hypercharge as well as the necessary component structure needed to couple  $\phi^+$  and  $\phi^0$  to  $u$  in the same way as  $d$  and  $e$ .

The scalar self-coupling parameters  $\mu$  and  $\lambda$  are unconstrained in principle. One would expect  $\mu$  to function as a mass for the field, but note that the term has opposite the expected sign (assuming  $\mu^2 > 0$ ); this subtlety has profound implications for the potential of  $\phi$ , as I will discuss in the next section.

### 2.1.3 Electroweak Symmetry Breaking and the Broken Lagrangian

Experimentally, matter fermions and weak gauge bosons are known to have mass, yet I gave no explicit mass terms in the Lagrangian, as stated in eqs. (2.3)-(2.5). In fact, it is not hard to convince oneself that (a) a mass term like  $M^2 A^\mu A_\mu$  for a gauge boson breaks its gauge symmetry, and (b) a Dirac mass term like  $m(\bar{\psi}_L \psi_R + \text{h.c.})$  for a fermion is intractable in light of the inequivalent electroweak quantum numbers ( $T^3$  and  $Y_w$ ) for left- and right-handed fields. It *is* completely tractable however to generate *effective* mass terms for both gauge bosons and fermions using a dynamic scalar field with the appropriate characteristics. This is the role of the Higgs field in the SM; the details of the emergence of these masses through the Higgs mechanism are as follows.

From a classical perspective, one can view the final two terms in eq. (2.5), which describe the self-interaction of the Higgs field, as a scalar potential<sup>4</sup>

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2. \quad (2.6)$$

In the alternate case where the  $\mu^2$  term is instead positive, this potential has a single minimum at  $\phi_0 = 0$ ; however, for a negative  $\mu^2$  term and appropriate related values for  $\mu$  and  $\lambda$ ,  $V$  has the shape seen in Figure 2.1. This potential is seen to have a continuously degenerate minimum, with a constant magnitude  $\phi_0 = \mu/\sqrt{2\lambda} \equiv v$  but arbitrary phase.

From the perspective of quantum field theory, this nonvanishing minimum corresponds to a *vacuum expectation value* (vev)  $\langle \phi \rangle$  for the scalar field  $\phi$ ; however, a field with such a vev cannot be quantized in the usual manner using creation/annihilation operators, which demands  $\hat{a}|0\rangle = 0$ ; yet, there is a simple way to bypass the issue: one can reparametrize the Higgs doublet given in eq. (2.2) as

$$\phi(x) = \begin{pmatrix} 0 \\ v + h^0(x) \end{pmatrix}, \quad (2.7)$$

where the dynamical real scalar field  $h^0(x)$  can be quantized as usual and treated as fluctuations about the nonvanishing but constant vacuum  $v$ ; an excitation of the field  $h^0$  is the Higgs boson. The alignment of  $v$  with the  $\phi^0$ -direction can be accomplished

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<sup>4</sup>an additional symmetry  $\phi \rightarrow -\phi$  is imposed on the Higgs Lagrangian to guarantee the presence of a stable minimum.

without loss of generality through a global  $SU(2)_L$  transformation; the complex scalar field  $\phi^+$  and the imaginary part of  $\phi^0$  have been set to zero using  $SU(2)_L \times U(1)_Y$  gauge transformations, and thus can be taken as unphysical. The above construction explicitly breaks the  $SU(2)_L \times U(1)_Y$  symmetry of the theory. Substituting this parametrization for  $\phi$  into eq. (2.5), one finds masses proportional to  $v$  have emerged for the fermions as a result of the breaking:

$$\mathcal{L}_{\mathcal{SM}} \ni -y_u^{ij} v \bar{u}_L^i u_R^j - y_d^{ij} v \bar{d}_L^i d_R^j - y_e^{ij} v \bar{e}_L^i e_R^j + \text{h.c.} \quad (2.8)$$

The same substitution in the final line of eq. (2.4) yields analogous terms for the gauge bosons, albeit with the presence of non-trivial mixing among the massless fields:

$$\mathcal{L}_{\mathcal{SM}} \ni \frac{v^2}{4} \left[ g^2 (W_1^\mu + iW_2^\mu) (W_\mu^1 - iW_\mu^2) + (-gW_\mu^3 + g'B_\mu)^2 \right]. \quad (2.9)$$

The combinations  $W_\mu^1 \mp iW_\mu^2 \equiv \sqrt{2} W_\mu^\pm$  used here were chosen by our forefathers because the coupling of  $W_\mu^{1,2}$  to matter consistently appears in these pairings, as one can see through the expansion of the  $q$ ,  $\ell$ , and  $\phi$  terms in eq. (2.4); since  $W_+^\mu W_\mu^- = (W_\mu^1)^2 + (W_\mu^2)^2$ , the mass eigenstates are equivalent. In contrast to that, the combination  $-gW_\mu^3 + g'B_\mu$  appears as a result of the diagonality of both the  $T^3$  and  $Y$  generators and cannot be avoided. Rather than ponder the curious cross terms, one can view the combination as a change of basis needed to describe the mass eigenstates manifestly. In fact, these mixed states correspond to the *physical* particles observed in experiment; yet, there were four bosons in the system prior to the breaking, so where has the fourth state gone? Let me define the (properly normalized) mixed  $W^3 + B$  state discussed above as

$$Z_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (gW_\mu^3 - g'B_\mu),$$

and also introduce the angle  $\theta_W$  such that  $\tan \theta_W = g'/g$ , so that  $Z_\mu = \cos \theta_W W_\mu^3 - \sin \theta_W B_\mu$ . Then there should exist a state

$$A_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (g'W_\mu^3 + gB_\mu) = \sin \theta_W W_\mu^3 + \cos \theta_W B_\mu,$$

orthogonal to  $Z_\mu$ , which is also a result of the rotation by  $\theta_W$ , and which apparently corresponds to the generator  $T^3 + Y$ ; if I write this generator as an  $SU(2)$  element acting on the Higgs doublet (recall  $Y_\phi = +1/2$ ), one can see that it annihilates the

vacuum in spite of the vev:

$$\langle 0 | (T^3 + Y) \phi | 0 \rangle = \frac{1}{2} \langle 0 | (\sigma^3 + \mathbb{I}) \phi | 0 \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = 0;$$

hence,  $T^3 + Y$  generates an unbroken symmetry, whose corresponding boson  $A_\mu$  remains massless. As the generator is diagonal, the unbroken symmetry is a  $U(1)$ , albeit a different one from that of weak hypercharge. One can easily be convinced that this symmetry corresponds to electromagnetism, with  $A_\mu$  as the photon and the electric charge as  $Q \equiv T^3 + Y$ .

In addition to the terms in eqs. (2.8) and (2.9), there is an otherwise identical set of terms with  $v \rightarrow h^0$  that give the interactions of the massive fermions (excluding the neutrino) and the gauge bosons with the neutral Higgs boson.

The covariant derivative in terms of the boson mass eigenstates is

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} (T^+ W_\mu^+ + T^- W_\mu^-) - \frac{ig}{\cos \theta_W} (T^3 - Q \sin^2 \theta_W) Z_\mu - ie Q A_\mu,$$

where  $T^\pm \equiv \frac{1}{2}(T^1 \mp iT^2)$ , and  $e = g \sin \theta_W$  is the electromagnetic coupling. In light of this derivative one finds chiral *charged currents*

$$\mathcal{L}_{\mathcal{SM}} \ni \frac{g}{\sqrt{2}} (\bar{u}_L^i \gamma^\mu V_{ckm}^{ij} d_L^j + \bar{\nu}_L^i \gamma^\mu e_L^i) W_\mu^+ + \text{h.c.} \quad (2.10)$$

chiral *neutral currents*

$$\mathcal{L}_{\mathcal{SM}} \ni \sum_{fLR} \frac{g}{\cos \theta_W} \bar{f}^i \gamma^\mu (T^3 - Q_f \sin^2 \theta_W) f^i Z_\mu, \quad (2.11)$$

where the sum is over both chiralities of all four flavors of fermion excluding  $\nu_R$ ; and the electromagnetic currents, coupling to Dirac spinors,

$$\mathcal{L}_{\mathcal{SM}} \ni \left( \frac{2}{3} \bar{u}^i \gamma^\mu u^i - \frac{1}{3} \bar{d}^i \gamma^\mu d^i - \bar{e}^i \gamma^\mu e^i \right) e A_\mu^+. \quad (2.12)$$

Recall that  $T^3$  is  $+1/2$  for  $u_L$  and  $\nu_L$ ,  $-1/2$  for  $d_L$  and  $e_L$ , and zero otherwise.

Note the presence of the matrix  $V_{ckm}$  in the charged currents of the quarks. Like the bosons, mass eigenstates for the quarks are generally different than flavor eigenstates; for flavor eigenstates  $u'_i, d'_i$  and mass eigenstates  $u_i, d_i$ , the mixing is given by

the transformations

$$u_i = U_{ij}^u u'_j, \quad d_i = U_{ij}^d d'_j,$$

where  $U_{ij}^{u,d}$  are  $3 \times 3$  unitary matrices. Inserting these transformations into a neutral current, one finds that the factors cancel with each other due to Hermitian conjugation; in the charged current, however, the new factors differ in flavor, and the resulting contribution

$$V_{\text{ckm}} \equiv U_R^{u\dagger} U_L^d \quad (2.13)$$

does not vanish in general. In fact, experiments have found that  $V_{\text{ckm}}$  is slightly off diagonal, implying that its presence in nature is physical. The matrix is parametrized by three mixing angles (one for each pair of generations) and a single imaginary phase,<sup>5</sup> which induces  $CP$ -violation in the model

The same phenomenon does not occur with leptons in the model due to the masslessness of the neutrino; the single rotation matrix coming from the charged leptons can be absorbed into a field redefinition. That said, we know that neutrinos do in fact have differing flavor and mass eigenstates, as their oscillation between mass eigenstates has been measured by experiments [2]. The corresponding transformation

$$\nu_i = U_{\nu}^{ij} \nu'_j \equiv V_{\text{pmns}}^{ij} \nu'_j$$

again consists of three angles, but generally may have two additional phases, for a total of three, due to the suspected Majorana nature of the neutrino. The mixing among generations is quite large in general, and even approximately maximal for  $\theta_{23} \sim 45^\circ$ . In fact, the largest (by far) angle of the CKM matrix,  $\theta_{\text{ckm}}^{12} \sim 12^\circ$  is only about 50% larger than the *smallest* angle in the PMNS,  $\theta_{\text{pmns}}^{13} \sim 9^\circ$ . The phases of the PMNS matrix are yet to be precisely measured, so the nature of  $CP$ -violation there is not yet known.

Returning to the substitution of the redefined Higgs + vev into eq. (2.5), one also finds that the Higgs boson itself acquires a mass term (with the proper sign)  $m_h = 2v\sqrt{\lambda}$ . Note that if I had *not* made gauge transformations to remove the additional components of  $\phi$ , we would see that they show up as massless scalars in the new Lagrangian. These components are known as *Nambu-Goldstone bosons* and are a general feature of spontaneously-broken field theories. Upon closer inspection, one

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<sup>5</sup>Note that a general  $3 \times 3$  unitary matrix has six phases, but here, five of them can be absorbed into field redefinitions.

would find terms like

$$\mathcal{L}_{SM} \ni \frac{i}{2} g v (W_\mu^+ \partial^\mu \phi^- - W_\mu^- \partial_\mu \phi^+) - \frac{v}{2} \sqrt{g^2 + g'^2} Z_\mu \partial^\mu \eta, \quad (2.14)$$

where  $\eta$  is the imaginary part of  $h^0$ ; these rather bizarre terms imply the gauge bosons can “convert” into the Goldstone bosons through two-particle, momentum-dependent interactions. Further terms show that in the interactions of the Goldstones with fermions, the bosons “imitate” the gauge bosons in terms of the configurations of fields with which they interact. These features led to the interpretation that the Goldstones are “eaten” by the gauge bosons, effectively becoming the longitudinal degrees of freedom absent in the massless states. Any other gauge choice or interpretation of the Goldstone bosons further confirm that the states are otherwise unphysical.

## 2.2 Measurement and The Success of the Standard Model

At this point, I have introduced the basic structure of the model and the interactions that arise from it. Application of the model to real-world measurements is traditionally built upon Hamiltonian formalism. In particular, if one defines from the Lagrangian a *Hamiltonian*

$$H = \int d^3\mathbf{x} \mathcal{H} \quad \text{where} \quad \mathcal{H} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\Psi}_i} \dot{\Psi}_i - \mathcal{L} \quad \propto \hat{a}_i^\dagger \hat{a}_i$$

for any field  $\Psi_i$  in the model, then using any term  $\mathcal{H}_{\text{int}} \in \mathcal{H}$  describing an interaction of  $\Psi_i$  with other fields  $\Psi_j$ , one can define the *S-matrix element*  $\langle \mathbf{p}_k \mathbf{p}_l | S | \mathbf{p}_i \mathbf{p}_j \rangle$  for an interaction  $\Psi_i \Psi_j \rightarrow \Psi_k \Psi_l$  via the operator

$$S \equiv \lim_{t, t_0 \rightarrow \pm\infty} \mathcal{T} \left[ \exp \left( -i \int_{t_0}^t dt' H_{\text{int}}(t') \right) \right] = \mathcal{T} \left[ \exp \left( -i \int_{-\infty}^{\infty} d^4x \mathcal{H}_{\text{int}}(t) \right) \right].$$

This seemingly simple expression hides a great deal of complexity; first note that

$$H_{\text{int}}(t) = e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)},$$

where  $H_0$  is the free part of the Hamiltonian; furthermore, considering the series expansion of the exponential, the  $n$ th term in the series is

$$\begin{aligned} S^{(n)} &= (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H_{\text{int}}(t_1) \dots H_{\text{int}}(t_n) \\ &= \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \dots \int_{t_0}^t dt_n \mathcal{T} [H_{\text{int}}(t_1) \dots H_{\text{int}}(t_n)], \end{aligned}$$

where  $\mathcal{T}$  implies one must take the *time ordered product* of the  $H$  operators. If  $H_{\text{int}}$  is proportional to some small coupling constant  $g \ll 1$ , as is the case for QED and electroweak processes at low energies, then each term in the series will be much smaller than the previous, so that one can treat the calculation of  $\langle f | S | i \rangle$  perturbatively. This is an especially crucial point because, despite of the asymptotic shrinking of the terms, the full series is typically divergent; because of this, entirely different methods are needed in cases of strong coupling  $g \sim 1$ .

To further probe the  $S$ -matrix formalism, consider as an example the simple QED scattering process  $e^- e^- \rightarrow e^- e^-$ ; in this case,  $\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi$ , or equivalently,  $\mathcal{H}_{\text{int}} = -Qe\bar{\psi}\gamma^\mu\psi A_\mu$ , such as for any term from eq. (2.12). Figure 2.2 shows the expansion of the scattering process in terms of *Feynman diagrams*, which are in one-to-one correspondence with non-trivial terms in the  $S$ -operator expansion. The first such term of the series, known as the *tree-level* diagram, is typically straightforward to calculate; for some processes, it may also be a sufficient approximation to some low-energy measurement of the matrix element. Note that in this case, the tree-level diagram corresponds to the  $n = 2$  term in the series. Consider the pair of  $\mathcal{H}_{\text{int}}$  operators in that term; each of the two fields  $\psi \sim \hat{a}$  act on the two initial electron states to annihilate the incoming particles, each of the two fields  $\bar{\psi} \sim \hat{a}^\dagger$  act on the two final electron states to create the outgoing particles, and the photon fields  $A_\mu$  are *Wick contracted* with each other to create the propagator.

The second term in the expansion in Figure 2.2 (corresponding to the  $n = 4$  term in the series) reveals a deeper mathematical complication with  $S$ -matrix formalism. The loop in the diagram, composed of two fermionic electron propagators, carries an arbitrary momentum  $\ell$ , corresponding to an  $\int d^4\ell$  in the calculation, which must be taken over all possible values of  $\ell$   $(-\infty, \infty)$ . Fermionic propagators are  $\sim i/\not{p}$ , so dimensional analysis suggests the integral is quadratically divergent; these seemingly problematic loop factors are a general feature of “radiative corrections” in a quantum field theory, *i.e.*, the quantum corrections to tree-level interactions arising from higher-



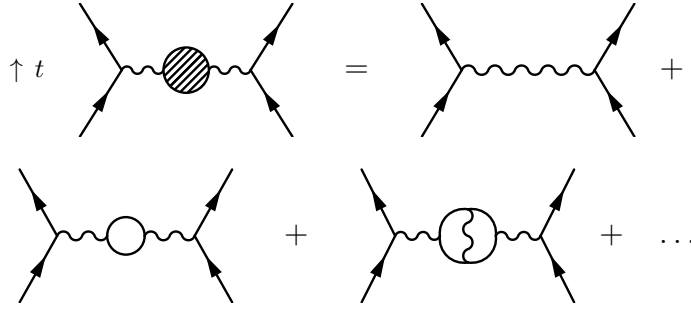


Figure 2.2: Feynman diagram expansion to third order of the  $S$ -matrix element for scattering of electrons by a photon.

order terms in the  $S$ -matrix. The apparent intractability can be handled using a clever and intricate technique called *renormalization* [25], which uses a *cut-off* energy scale or other *regulator* to quarantine the infinite part of the integral, then cancels that infinite part against counter-terms associated to each of the *bare* parameters of the theory, namely the masses, coupling constants, and wave-function normalization factors as they appear in the original Lagrangian. In doing a complete analysis of the renormalization of a particular theory, one finds not only that the cut-off (ultimately  $\rightarrow \infty$ ) is unphysical, but also that the physical values of the parameters of the theory generally vary with the overall energy scale of a measurement, and this variation is determined by the finite parts of the higher-order loop diagrams in the series expansion. The formalism describing this *running* of parameters with scale has a rich, group-like mathematical structure of its own [26, 27].

With confidence that, despite its superficial complications,  $S$ -matrix theory is mathematically valid, I can return to its use for calculating measurable features of the SM. The non-trivial part of the  $S$  operator can be extracted explicitly by writing  $S = \mathbf{1} + iT$ ; furthermore,  $T$  is related to the *Feynman amplitude*  $\mathcal{M}$ , generically known as the “matrix element”, by

$$\langle \mathbf{p}_k \mathbf{p}_l | iT | \mathbf{p}_i \mathbf{p}_j \rangle = (2\pi)^4 \delta^4(\Sigma p) i\mathcal{M},$$

where  $\delta^4(\Sigma p) = \delta^4(p_i + p_j - p_k - p_l)$  gives the total four-momentum conservation for the process. Since the Hamiltonian, whose eigenvalues are energy, is a Hermitian operator,  $S$  is a unitary operator; consequently, the absolute square of a  $T$ -matrix element gives the probability for the occurrence of the corresponding interaction if the following conditions are satisfied: (a) the free incoming particles are present at

$t \rightarrow -\infty, \mathbf{x} \rightarrow \infty$ , (b) the system undergoes eternal time evolution via the operator  $\exp(-iHt)$ , and (c) the free outgoing particles are present at  $t \rightarrow \infty, \mathbf{x} \rightarrow \infty$ . Using this prescription and the above definition for  $\langle f | iT | i \rangle$ , one can calculate the *scattering cross section*  $\sigma$  of the interaction  $\Psi_i \Psi_j \rightarrow \Psi_k \Psi_l$ :

$$\sigma = \frac{1}{4E_i E_j v} \int \frac{\mathbf{d}^3 \mathbf{p}_k}{(2\pi)^3 2E_k} \int \frac{\mathbf{d}^3 \mathbf{p}_l}{(2\pi)^3 2E_l} (2\pi)^4 \delta^4(\Sigma p) |\mathcal{M}|^2,$$

where  $v$  is the relative velocity of the incoming particles. A similar expression can be written for the *decay width* of a massive particle. One can make explicit measurements of a cross section or a decay width, represented by some  $S$ -matrix element, by observing the output of particle beams incident upon each other, so long as (a) the interaction occurs in relative isolation, at a “large” distance from the detectors, and (b) the output is observed a very large number of times, so as to simulate the eternality of the probabilities.

Indeed, precisely such measurements have been made for decades, at particle accelerator experiments such as the Tevatron, LEP, and now the LHC; every probability associated with an interaction predicted by the SM agrees with the experimental data to truly remarkable and unprecedented levels of precision. Furthermore, several of the particles of the SM were predicted to exist by the completed framework *prior to being observed*; the mass of each particle was accurately predicted as well. This was the case for the heavy quarks, the  $W$  and  $Z$  bosons, and, most recently, the Higgs boson  $h^0$ , which was not seen until 2012. The mass of the Higgs was perhaps a bit higher than originally expected, and so its observation had to wait for the construction of CERN’s Large Hadron Collider; yet, due to the extremely thorough record of prior successes of the model, physicists remained confident throughout the years that the Higgs boson would be seen.

The model also makes similarly remarkable predictions involving the precision of measured values related to the hydrogen atom, the magnetic moment of the electron, and other low-energy or atomic phenomena. These values were previously calculated in the context of non-relativistic quantum mechanics or classical electromagnetism and showed unexplained discrepancies with measurements; the discrepancies are largely eliminated when the analogous calculations are performed in the context of the SM.

## 2.3 The Limitations of the Model and a Need for New Physics

Despite the extreme robustness and precision of the Standard Model, it is at the same time a manifestly incomplete theory, and it leaves some number of mysteries unsolved. Some of the most obvious aspects of its incompleteness are:

- The model relies on the presence of roughly 19 parameters, including masses, coupling constants, and generational mixing parameters, whose values are known through measurement and are otherwise completely arbitrary; in some cases, the observed values are arguably fine-tuned. Such tunings include the more conceptual concern of the presence of the three generations of otherwise-identical fermions with different masses, where a unique and unexplained hierarchical mass spectrum exists for each flavor.
- The model predicts that neutrinos are massless, while there is ample experimental evidence otherwise. Freely propagating neutrinos are known to oscillate from one generation to another; the only known mechanism for such a process is through CKM-like mixing among flavor and mass eigenstates. Hence, neutrinos seem to have mass after all, however small those masses may be.
- The model makes no mention whatsoever of gravity; furthermore, it consequently gives no explanation for the presence of dark energy and no realistic explanation for the presence of dark matter.

In addition to these omissions, there are few more subtle peculiarities that suggest theoretical incompleteness:

- Like the parameters of the model, the internal gauge symmetry group of the SM is *ad hoc*, as it was originally determined primarily through phenomenological arguments.
- The negative scalar mass parameter and therefore the entirety of electroweak breaking is similarly arbitrary from the theoretical perspective; the Higgs mechanism was devised to solve the problem of giving mass to the particles and is not motivated by any aspect of the mathematical structure of the model.

- Radiative corrections to the Higgs propagator are quadratically dependent on the energy scale of the measurement; these strongly divergent contributions, which are unique to scalar fields, severely renormalize the mass of the particle. Naively, one would expect this to lead to arbitrarily large corrections to the mass, pushing it all the way up to the *Planck scale*, where gravitational effects become significant,  $M_{\text{Pl}} \sim 10^{18} \text{ GeV}$ . Yet, we see the Higgs boson to have a comparably minuscule mass of 126 GeV; the SM offers no explanation for this truly enormous discrepancy. This puzzle is known as the *hierarchy problem*.

These unsolved questions have led physicists to pursue a great number of ideas for the extension of the standard model, to varying degrees of success. So far, very little has been “officially” added to the theory, as no definitive experimental evidence has been observed in support of any hypothesis.

Soon after the completion of the SM framework in the early 1970s, a new class of models emerged from attempts to extend the notion of electroweak unification to more fundamental levels. It seemed that if electromagnetism and weak interactions were unified earlier in the universe, then perhaps *that* era followed from the breaking of *yet another* unification of the electroweak force with the strong force. This concept, known as *Grand Unified theory*, offered some relief to the arbitrariness of the SM gauge group. The first models were developed by Pati and Salam [4] and then Georgi and Glashow [3] in 1974. Further extensions of these models in turn led to the development of  $SO(10)$  unification, which will be a primary topic for the remainder of this work.

Taking a closer look at the Higgs mass corrections, one will notice that they arise from both bosonic and fermionic loops; furthermore, these contributions come with opposite signs. This subtlety led some physicists in the 1970s to propose a practical application of an otherwise-esoteric idea known as *supersymmetry*, which relates bosons to fermions through a subtle extension of spacetime itself. I will introduce this concept in more detail in the next chapter.

# Chapter 3

## Supersymmetry

Consider the diagrams for the one-loop corrections to the Higgs boson mass squared parameter  $m_h^2$  seen in Figure 3.1; the correction from a generic fermion  $f$  in (a) can be written as

$$\Delta m_h^2 = -\frac{y_f^2}{8\pi^2}\Lambda_{\text{UV}}^2 + \dots, \quad (3.1)$$

where  $\Lambda_{\text{UV}}$  is the cutoff energy scale used to regulate the loop integral for renormalization; the analogous contribution from a generic scalar  $S$ , seen in Figure 3.1(b) is

$$\Delta m_h^2 = \frac{\lambda_S}{16\pi^2}\Lambda_{\text{UV}}^2 + \dots \quad (3.2)$$

The terms in “...” are at most logarithmically dependent on  $\Lambda_{\text{UV}}$ . Assuming no additional physics aside from gravity, the cutoff is at the Planck scale, and these corrections are at least 25 orders of magnitude larger than the physical value of  $(126 \text{ GeV})^2$ , depending on the size of the coupling constants. Naively, this suggests a staggeringly large cancellation between the bare Higgs mass  $m_h$  and these corrections. Note that the contributions from the log-divergent terms are a much more natural  $\mathcal{O}(m_h^2)$ .

If instead one requires the  $\Lambda_{\text{UV}}^2$  corrections to be similarly natural, then one finds a need for  $\Lambda_{\text{UV}} \lesssim \mathcal{O}(1) \text{ TeV}$ , which naively suggests a need for new physics at that scale.

There is, however, a more creative solution one might consider. Since the correction from the fermion is negative but the correction from the scalar is positive, under the restriction that  $y_f^2 = \lambda_S$ , then a theory with two such bosons for each fermion would have a cancellation of these problematic terms against each other; in fact the cancellation would persist to all orders. Since these are interactions with the Higgs field, the above restriction on the couplings corresponds to the scalar and the fermion

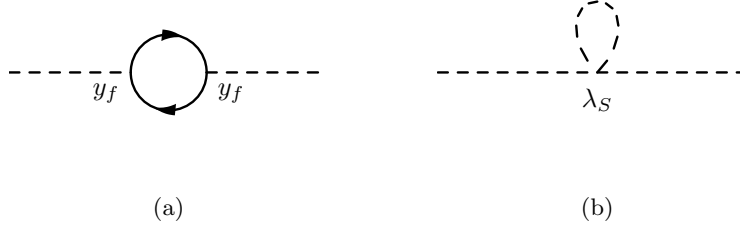


Figure 3.1: One-loop diagrams from (a) a fermion  $f$  and (b) a scalar  $S$  for the Higgs propagator, which give corrections to the bare mass squared parameter  $m_h^2$ .

having identical masses.

It turns out that such a theory does exist. *Supersymmetry* employs fermionic operators to enable transformation of bosons into fermions, and vice versa, through a subtle extension of spacetime itself. The formalism was discovered in the early 1970s and was explored for mainly novel reasons until the realization of its application to the hierarchy problem discussed above. This chapter will introduce the basic structure of supersymmetry (SUSY) and give the form of a realistic extension of the standard model that utilizes the concept to address not only the hierarchy problem, but also several other aspects of the puzzles of the SM.

### 3.1 Basic Supersymmetry Formalism

Consider a bosonic state  $|b\rangle$  and a fermionic state  $|f\rangle$ . The generator of supersymmetry is a fermionic operator  $\hat{Q}$  such that  $\hat{Q}|b\rangle \sim |f\rangle$  and  $\hat{Q}|f\rangle \sim |b\rangle$ . In general, a proper supersymmetric transformation trades a bosonic degree of freedom for a fermionic one in a one-to-one manner. More realistically, one can understand the Weyl spinor operators  $\hat{Q}^\alpha$  and  $\hat{Q}_{\dot{\alpha}}^\dagger$  as a peculiar extension of the Poincaré algebra such that

$$\{\hat{Q}_\alpha, \hat{Q}_{\dot{\alpha}}^\dagger\} = -2\sigma_{\alpha\dot{\alpha}}^\mu \hat{P}_\mu, \quad (3.3)$$

and

$$\{\hat{Q}_\alpha, \hat{Q}_\beta\} = 0; \quad \{\hat{Q}_{\dot{\alpha}}^\dagger, \hat{Q}_{\dot{\beta}}^\dagger\} = 0, \quad (3.4)$$

where  $\hat{P}_\mu = i\partial_\mu$  is the generator of momentum and  $\sigma_{\alpha\dot{\alpha}}^\mu$  is the usual extension of the Pauli matrices ( $\mathbb{I}, \vec{\sigma}$ ), except I have written the  $SL(2, \mathbb{C})$  spinor space indices explicitly.

The indices of  $Q^\alpha$  (and  $\sigma_{\alpha\dot{\alpha}}^\mu$ ) are raised and lowered using the Levi-Civita tensor  $\epsilon^{\alpha\beta}$ , with  $\epsilon^{12} = -\epsilon_{12} = 1$ . Note also that

$$\left[ \hat{Q}_\alpha, \hat{P}_\mu \right] = 0; \quad \left[ \hat{Q}_\alpha^\dagger, \hat{P}_\mu \right] = 0 \quad (3.5)$$

*i.e.*, supersymmetric transformations commute with all translations, implying that a boson and a fermion transforming into one another under SUSY will have the same mass. The above relations comprise a closed extension of the Poincaré algebra, forming what is known as a *graded algebra* or a *superalgebra*. This supersymmetric loophole is the only exception to the Coleman-Mandula “no-go” theorem, which implies that the only symmetry group of the  $S$ -matrix consistent with QFT is a direct product of Poincaré and some internal compact Lie group.

Since we have not seen *superpartner* particles for the light SM particles in nature, it would seem that SUSY is broken symmetry at low energies; however, in order to preserve the perfect cancellations in the Higgs mass corrections, which requires that  $y_f^2 = \lambda_S$  still holds in the broken theory, the breaking of SUSY must be isolated from the dynamics. This prescription is known as *soft breaking* of the theory, and it is realized mainly through (positive) mass terms for the superpartners, which may be the result of some “hidden sector” physics, cut off from the low energy physics, but are otherwise free parameters. I will discuss this concept and its implications in more detail shortly.

### 3.1.1 Constructing a Supersymmetric Model

<sup>1</sup> The most basic non-trivial SUSY model one can construct involves a free single Weyl fermion  $\psi = \psi_\alpha$  and its two free scalar superpartners, which are conventionally treated as one complex field  $\phi = (A + iB)/\sqrt{2}$ . Note that for a realistic model with both matter fermions and scalar bosons, each type of field will have the other type as its superpartner; hence, I will keep this discussion very general so it can apply to either case. The supersymmetric transformation of a field is defined as

$$-i\sqrt{2}\delta(\varepsilon)X \equiv \left[ \varepsilon\hat{Q} + \varepsilon^\dagger\hat{Q}^\dagger, X \right] \quad (3.6)$$

for any field  $X$  and infinitesimal parameter  $\varepsilon_\alpha$ , which is a constant Grassmann (anti-commuting) spinor; the contraction  $\varepsilon\hat{Q} \equiv \epsilon^{\alpha\beta}\varepsilon_\alpha\hat{Q}_\beta$ , and  $\varepsilon^\dagger\hat{Q}^\dagger$  is analogous. One may

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<sup>1</sup>This discussion largely follows that of ref. [28]; please see that work for further detail.

expect that the corresponding supersymmetric Lagrangian is simply

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi + i \psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi, \quad (3.7)$$

where  $\psi^\dagger \bar{\sigma}^\mu \psi \equiv \psi_\alpha^\dagger (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \psi_\alpha$ . At first glance, this will seem correct: the transformations of the fields are

$$\begin{aligned} \delta(\varepsilon) \phi &= \varepsilon \psi, & \delta(\varepsilon) \phi^* &= \varepsilon^\dagger \psi^\dagger, \\ \delta(\varepsilon) \psi_\alpha &= i (\sigma^\mu \varepsilon^\dagger)_\alpha \partial_\mu \phi, & \delta(\varepsilon) \psi_\alpha^\dagger &= -i (\varepsilon \sigma^\mu)_{\dot{\alpha}} \partial_\mu \phi^*, \end{aligned} \quad (3.8)$$

where, *e.g.*,  $(\sigma^\mu \varepsilon^\dagger)_\alpha \equiv \sigma_{\alpha\dot{\alpha}}^\mu \varepsilon^{\dot{\alpha}}$ ; utilizing these transformations in eq. (3.7), one finds that

$$\begin{aligned} \delta \mathcal{L}_\phi &= \varepsilon \partial^\mu \psi \partial_\mu \phi^* + \varepsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi, \\ \delta \mathcal{L}_\psi &= -\varepsilon \partial^\mu \psi \partial_\mu \phi^* - \varepsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi + \text{total derivatives}; \end{aligned} \quad (3.9)$$

since the total derivative vanishes in the action,  $\mathcal{L}$  is in fact invariant under a SUSY transformation.

Still, though, one must address the closure of the superalgebra. Considering successive transformations  $[\delta(\varepsilon_2), \delta(\varepsilon_1)]X$ , one sees that

$$\begin{aligned} [\delta(\varepsilon_2), \delta(\varepsilon_1)] \phi &= i \left( \varepsilon_2 \sigma^\mu \varepsilon_1^\dagger - \varepsilon_1 \sigma^\mu \varepsilon_2^\dagger \right) \partial_\mu \phi, \\ [\delta(\varepsilon_2), \delta(\varepsilon_1)] \psi_\alpha &= i \left( \varepsilon_2 \sigma^\mu \varepsilon_1^\dagger - \varepsilon_1 \sigma^\mu \varepsilon_2^\dagger \right) \partial_\mu \psi_\alpha \\ &\quad + i \varepsilon_{1\alpha} \varepsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi - i \varepsilon_{2\alpha} \varepsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi. \end{aligned} \quad (3.10)$$

For the scalar field, a product of SUSY transformations returns a derivative of the field, as suggested by eq. (3.3). The fermion case is similar once one notes that the two extra terms in the transformation will vanish on-shell, when the classical equation of motion  $\bar{\sigma}^\mu \partial_\mu \psi = 0$  holds. This is something, but it is not enough to build a truly consistent supersymmetric quantum model.

This problem can be resolved by introducing an *auxiliary field* into the system with the right properties. The field  $F$  will be a complex scalar with  $[F] = 2$ , and the contribution to the Lagrangian is

$$\mathcal{L}_F = -F^* F; \quad (3.11)$$



the field has a non-dynamical, algebraic equation of motion, and so should be treated as unphysical. The field transforms under SUSY as

$$\delta(\varepsilon)F = i\varepsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi, \quad \delta(\varepsilon)F^* = -i\partial_\mu \psi^\dagger \bar{\sigma}^\mu \varepsilon; \quad (3.12)$$

combining this with an augmentation of the fermion transformations,

$$\delta(\varepsilon)\psi_\alpha = i(\sigma^\mu \varepsilon^\dagger)_\alpha \partial_\mu \phi + \varepsilon_\alpha F, \quad \delta(\varepsilon)\psi_\alpha^\dagger = -i(\varepsilon \sigma^\mu)_{\dot{\alpha}} \partial_\mu \phi^* + \varepsilon_{\dot{\alpha}}^\dagger F^*, \quad (3.13)$$

gives the desired off-shell closure of the complete system.

Therefore, eq. (3.7) together with eq. (3.11) give a complete supersymmetric Lagrangian for a free scalar, its fermionic superpartner, and the corresponding auxiliary field, which is known as the *Wess-Zumino* model of supersymmetry; it will be the basis for building a realistic model of SUSY-invariant interactions.

### Yukawa Interactions and the Superpotential

To introduce interactions in the model, I first define the *superpotential*  $W$ :

$$W \equiv \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k, \quad (3.14)$$

where the indices  $i, j, k$  generically run over any flavor quantum numbers. Note that  $W$  is *holomorphic*, *i.e.*, analytic in  $\phi$ , and completely symmetric under exchange of indices. Now I can write

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}W^{ij}\psi_i\psi_j + W^i F_i + \text{h.c.s}, \quad (3.15)$$

where

$$\begin{aligned} W^{ij} &\equiv \frac{\delta^2 W}{\delta\phi_i \delta\phi_j} = M^{ij} + y^{ijk}\phi_k \text{ and} \\ W^i &\equiv \frac{\delta W}{\delta\phi_i} = M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k. \end{aligned} \quad (3.16)$$

The  $F$ -terms in the full Lagrangian lead to the algebraic equations of motion

$$F_i = -W_i^* \quad \text{and} \quad F^{*i} = -W^i,$$

which I can utilize to rewrite the interaction Lagrangian as

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}W^{ij}\psi_i\psi_j + \text{h.c.} - V(\phi, \phi^*), \quad (3.17)$$

where

$$V(\phi, \phi^*) \equiv W^i W_i^* = \left| \frac{\delta W}{\delta \phi_i} \right|^2 \quad (3.18)$$

is the *scalar potential* for the system, giving the usual mass, cubic, and quartic terms for the scalar field(s)  $\phi$ ; similarly, the  $W^{ij}$  term gives a (holomorphic) fermion mass term and Yukawa coupling with the scalar partner  $\phi$ .

### Gauge Fields and Interactions

To expand a Wess-Zumino-type model to include gauge interactions, I will first need to consider the supersymmetric transformation of gauge bosons. Like a scalar field, spin-1 fields will also have fermionic spin-1/2 superpartners. For a gauge field  $A_\mu^a$ , I will denote the “gaugino” superpartner as  $\lambda_\alpha^a$ .<sup>2</sup> The Lagrangian for the gauge sector is then

$$\mathcal{L}_g = -\frac{1}{4}A_a^{\mu\nu}A_{\mu\nu}^a - i\lambda^{\dagger a}\bar{\sigma}^\mu D_\mu\lambda^a + \frac{1}{2}D^a D^a; \quad (3.19)$$

$D^a$  is, like  $F$ , an auxiliary field that allows the superalgebra to close off-shell; unlike  $F$ , however, it is a real field (since the on-shell boson has only one additional degree of freedom).  $F_{\mu\nu}^a$  is defined in the usual manner (*e.g.*, as seen in the previous chapter), and the covariant derivative acts on the gaugino as  $D_\mu\lambda^a = \partial_\mu\lambda^a + gf^{abc}A_\mu^b\lambda^c$ . Both  $D^a$  and  $\lambda^a$  transform in the adjoint representation of the gauge group. The supersymmetric transformations of the fields are

$$\begin{aligned} \delta(\varepsilon)A_\mu^a &= -\frac{1}{\sqrt{2}}(\varepsilon^\dagger\bar{\sigma}^\mu\lambda^a + \text{h.c.}), \\ \delta(\varepsilon)\lambda_\alpha^a &= \frac{i}{2\sqrt{2}}(\sigma^\mu\bar{\sigma}^\nu\varepsilon)_\alpha A_{\mu\nu}^a + \frac{1}{\sqrt{2}}\varepsilon_\alpha D^a, \\ \delta(\varepsilon)D^a &= \frac{i}{\sqrt{2}}(\varepsilon^\dagger\bar{\sigma}^\mu D_\mu\lambda^a - D_\mu\lambda^{\dagger a}\bar{\sigma}^\mu\varepsilon). \end{aligned} \quad (3.20)$$

One couples the fermions  $\psi$  and scalars  $\phi$  to  $A_\mu^a$  through the usual promotion of

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<sup>2</sup>Note that in four-component bispinor notation, the gaugino is a *Majorana* fermion, meaning  $\psi^c = \psi$

the derivative  $\partial_\mu \rightarrow D_\mu$  in the Lagrangian eq. (3.7); however, one must also consider allowed fermion-boson-gaugino interactions, which are of the form

$$\mathcal{L}_{\text{g,int}} = -g\sqrt{2}(\phi^* t^a \psi \lambda^a + \text{h.c.}) + g \phi^* t^a \phi D^a, \quad (3.21)$$

where  $t^a$  are the generators of the gauge group. As with  $F$ , one can again use the algebraic equation of motion for the auxiliary field  $D^a = -g\phi^* t^a \phi$  to eliminate it from the Lagrangian. This also results in an additional contribution to the scalar potential,

$$V(\phi, \phi^*) \equiv W^i W_i^* + \frac{1}{2} g^2 (\phi^* t^a \phi)^2. \quad (3.22)$$

Note this can also be written as  $V = |F|^2 + \frac{1}{2} D^2$ , which gives rise to the common nomenclature “F-term” and “D-term” when referring to the two scalar potential contributions. Note that in the presence of multiple gauge groups (as in the SM), one finds a simple sum of contributions from each.

To guarantee invariance of the entire interacting model under SUSY transformations, one must replace the derivatives in the transformations  $\delta\psi$  and  $\delta F$  with gauge covariant derivatives, and augment the transformation of  $F$  by the inclusion of a term involving the gaugino

$$\delta(\varepsilon)F_i = i\varepsilon^\dagger \bar{\sigma}^\mu D_\mu \psi_i - g\sqrt{2}(t^a \phi)_i \varepsilon^\dagger \lambda^a \quad (3.23)$$

and similar for  $F^{*i}$ . Now the entire system is invariant (up to total derivatives) under the transformations given by eqs. (3.20), the gauge covariant versions of (3.8), and the above transformation for  $F$ .

## Soft Supersymmetry Breaking

As mentioned previously, the absence of superpartners in nature suggests that SUSY is a broken symmetry. One would like to find that the symmetry is broken spontaneously, like that of electroweak theory; early on, the possibilities of taking  $\langle F \rangle \neq 0$  [29] or  $\langle D \rangle \neq 0$  [30] were explored thoroughly; both options can be implemented in general SUSY models to break the symmetry, but in the context of supersymmetric extension of the standard model, both methods fail to give a realistic mass spectrum for the superpartners. In the end, one is left to consider *soft breaking* of SUSY through terms with couplings of explicitly positive mass dimension.

Soft breaking terms allowed in the general interacting model described above are

$$\mathcal{L}_{\text{soft}} = -\frac{1}{2}M_g\lambda^a\lambda^a - \frac{1}{2}b^{ij}\phi_i\phi_j - \frac{1}{6}a^{ijk}\phi_i\phi_j\phi_k + \text{c.c.s} - (m^2)^i_j\phi^{*j}\phi_i. \quad (3.24)$$

We will see more about the consequences of these terms in the context of the Minimally Supersymmetric SM in the next section. I will also discuss briefly some mechanisms that could dynamically give rise to these terms.

### 3.1.2 Superfields

In order to make supersymmetry manifest in a field theory, one needs to consider *superfields*, or multiplets containing a field and its superpartner. In order to accommodate the fundamental spacetime differences between bosons and fermions in the same object, one needs to expand the spacetime itself to include four new fermionic coordinates  $x^\mu \rightarrow (x^\mu, \theta^\alpha, \theta^\dagger_{\dot{\alpha}})$ . These new coordinates of dimension  $[\theta] = -\frac{1}{2}$  commute with  $x^\mu$  but anti-commute with themselves and each other. Products or contractions of thetas are generally the same as those for any Weyl fermions, but note also that  $\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta$  for identical spinors.

The Grassmann nature of the thetas has the peculiar implication that the square of any individual component vanishes,  $(\theta_1)^2 = (\theta_2)^2 = 0$ . As a result, any general function of  $\theta$  and  $\theta^\dagger$  can be written as a terminating series. Therefore, the most general superfield  $\mathcal{S}$  one can write has the form

$$\begin{aligned} \mathcal{S}(x^\mu, \theta, \theta^\dagger) = & a + \theta\chi + \theta^\dagger\xi^\dagger + \theta^2b + (\theta^\dagger)^2c + \theta^\dagger\bar{\sigma}^\mu\theta v_\mu \\ & + (\theta^\dagger)^2\theta\eta + \theta^2\theta^\dagger\zeta^\dagger + (\theta^\dagger)^2(\theta)^2d, \end{aligned} \quad (3.25)$$

where all component fields are functions of spacetime. When comparing to the fields in the previous section, one can determine that  $a$  is scalar-like,  $\chi, \xi$  is fermion-like,  $\eta, \zeta$  gaugino-like, and  $b, c, d$  auxiliary-field-like. The complex scalar component fields  $a, b, c, d$  give eight real bosonic degrees of freedom,  $v_\mu$  gives eight more as a complex vector field, and the (always complex) Weyl fermion components  $\chi, \xi^\dagger, \eta, \zeta^\dagger$  give sixteen fermionic degrees of freedom.  $\mathcal{S}$  transforms under general SUSY transformations as a

translation in superspace,

$$\begin{aligned}\mathcal{S}(x^\mu, \theta, \theta^\dagger) &\rightarrow \exp \left[ i(\varepsilon Q + \varepsilon^\dagger Q^\dagger) \right] \mathcal{S}(x^\mu, \theta, \theta^\dagger) \\ &= \mathcal{S} \left( x^\mu - i\varepsilon \sigma^\mu \theta^\dagger + i\theta \sigma^\mu \varepsilon^\dagger, \theta + \varepsilon, \theta^\dagger + \varepsilon^\dagger \right);\end{aligned}$$

note that superfields are closed under multiplication, which is a crucial factor for constructing Lagrangians.

We can write the SUSY generators as differential operators in superspace:

$$Q_\alpha = -i \frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu \theta^\dagger)_\alpha \partial_\mu; \quad Q^\dagger_{\dot{\alpha}} = i \frac{\partial}{\partial (\theta^\dagger)^{\dot{\alpha}}} + (\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu; \quad (3.26)$$

Using these operators, one can show that supersymmetric transformations written in terms of these differential operators are equivalent to the transformations in terms of the quantum operators as seen in eq. (3.6):

$$\left[ \varepsilon \hat{Q} + \varepsilon^\dagger \hat{Q}^\dagger, X \right] = (\varepsilon Q + \varepsilon^\dagger Q^\dagger) X$$

for any superfield component  $X$ . One can also define the *chiral covariant derivatives*

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i (\sigma^\mu \theta^\dagger)_\alpha \partial_\mu; \quad D^\dagger_{\dot{\alpha}} = -\frac{\partial}{\partial (\theta^\dagger)^{\dot{\alpha}}} - i (\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu, \quad (3.27)$$

such that  $\delta(\varepsilon)(D_\alpha \mathcal{S}) = D_\alpha(\delta(\varepsilon)\mathcal{S})$ , and similar for  $D^\dagger_{\dot{\alpha}}$ . Note that these operators satisfy the same superalgebra as, and also anti-commute with,  $Q$  and  $Q^\dagger$ .

## Irreducible Supermultiplets

The general superfield  $\mathcal{S}$  is a reducible representation in the superalgebra space. This is perhaps evident in light of the independent supersymmetric closure of *each* of the sets of fields  $\{\phi, \psi, F\}$  and  $\{A, \lambda, D\}$ , as seen in the previous section. One can obtain the desired irreducible multiplets by constraining  $\mathcal{S}$  in specific ways.

The *chiral* or *left-chiral superfield*  $\Phi_L$ , which generically corresponds to an irreducible supermultiplet containing a matter fermion or scalar boson, arises from the constraint equation

$$D^\dagger_{\dot{\alpha}} \Phi_L = 0. \quad (3.28)$$

Using the convenient change of variables  $y^\mu \equiv x^\mu + i\theta \sigma^\mu \theta^\dagger$ , one can write a general

chiral superfield as

$$\Phi_L(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y); \quad (3.29)$$

where the component fields  $\{\phi, \psi, F\}$  correspond to those from the previous section. Note one can quickly determine that a chiral superfield has  $[\Phi] = 1$ .

Similarly, the *anti-chiral* or *right-chiral superfield*  $\Phi_R^*$  is the complex conjugate of  $\Phi_L$  and arises from the constraint equation

$$D_\alpha \Phi_R^* = 0; \quad (3.30)$$

Using the corresponding change of variables  $y^{\mu*} \equiv x^\mu - i\theta\sigma^\mu\theta^\dagger$ , one can write a general anti-chiral superfield as

$$\Phi_R^*(y^*, \theta^\dagger) = \phi^*(y^*) + \sqrt{2}\theta^\dagger\psi^\dagger(y^*) + (\theta^\dagger)^2 F^*(y^*). \quad (3.31)$$

Finally, the *vector superfield*  $\mathcal{A}$ , which is the irreducible supermultiplet containing a gauge boson field, is obtained by demanding the superfield is real, *i.e.*, by imposing the condition  $\mathcal{S} = \mathcal{S}^*$ . Comparing with eq. (3.25), this implies

$$a = a^*, \quad \chi = \xi, \quad b = c^*, \quad v_\mu = v_\mu^*, \quad \eta = \zeta, \quad d = d^*.$$

Note that the combinations of chiral/anti-chiral superfields  $\Phi^*\Phi$ ,  $\Phi + \Phi^*$ , and  $i(\Phi^* - \Phi)$  are also real and hence are vector superfields.

We can write the generalization of an infinitesimal gauge transformation to supersymmetric form as

$$\mathcal{A}^a \rightarrow \mathcal{A}^a + i(\Omega^{*a} - \Omega^a) + gf_{abc}\mathcal{A}^b(\Omega^{*c} + \Omega^c) \quad (3.32)$$

for some chiral superfield gauge transformation parameter  $\Omega$ ; the expression simplifies in the usual manner for Abelian symmetry. Such a transformation will yield the proper form for a gauge transformation of the gauge boson field, as well as the proper transformations for the gaugino  $\lambda^a$  and auxiliary field  $D^a$  for non-Abelian cases. Using a convenient supergauge choice  $\Omega^* = -\Omega$ , known as the *Wess-Zumino gauge*, one can write a vector superfield in the form

$$\mathcal{A}^a(x^\mu, \theta, \theta^\dagger) = \theta^\dagger \bar{\sigma}^\mu \theta A_\mu^a + (\theta^\dagger)^2 \theta \lambda^a + \theta^2 \theta^\dagger \lambda^{\dagger a} + \frac{1}{2}(\theta^\dagger)^2 \theta^2 D^a, \quad (3.33)$$

where the component fields  $\{A, \lambda, D\}$  correspond to those for a supersymmetric gauge model from the previous section. In this form, it is apparent that  $[\mathcal{A}] = 0$ .

All three types of superfields discussed above close independently under multiplication.

## A Complete Superfield Lagrangian

Using the superfield notation from the previous subsection and the details introduced in Section 3.1.1, one can write a complete supersymmetric action in terms of integrals of superfields in superspace. One might see the final form as rather unexpected, in that it relies on several unusual intermediate results.

First I need to discuss how one performs Grassmann integration. Using these basic rules,

$$\int d\theta^\alpha = 0, \quad \int d\theta^\alpha \theta^\beta = \delta^{\alpha\beta},$$

and noting that  $d^2\theta = -\frac{1}{4}\epsilon_{\alpha\beta}d\theta^\alpha d\theta^\beta$ , one can see that the integration of a function  $f(\theta, \theta^\dagger)$  over some measure in superspace picks out the coefficient in  $f$  of the term with theta dependence matching that of the signature; *e.g.*,

$$\begin{aligned} \int d^2\theta \mathcal{S} &= b + \theta^\dagger \zeta^\dagger + (\theta^\dagger)^2 d, \\ \int d^2\theta d^2\theta^\dagger \mathcal{S} &= d, \quad \text{etc.} \end{aligned}$$

Now, I can use the above principle to build my superfield Lagrangian by integrating certain products of superfields over certain portions of superspace. For instance, in the expansion of the superfield product  $\Phi_R^* \Phi_L$ , one will find that the “D-term”  $\sim (\theta^\dagger)^2 \theta^2$  precisely gives the free Wess-Zumino Lagrangian seen in eqs. (3.7) and (3.11):

$$[\Phi^* \Phi]_D \equiv \int d^2\theta d^2\theta^\dagger \Phi^* \Phi = \partial^\mu \phi^* \partial_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - F^* F + \partial_\mu(\dots); \quad (3.34)$$

similarly, if I reconsider the concept of the Wess-Zumino superpotential  $W(\phi)$  in the context of superfields, *i.e.*,

$$W(\Phi) \equiv \frac{1}{2} M^{ij} \Phi_i \Phi_j + \frac{1}{6} y^{ijk} \Phi_i \Phi_j \Phi_k, \quad (3.35)$$

one finds that the “F-terms”  $\sim \theta^2$  for  $W(\Phi)$  and  $W(\Phi^*)$  together give

$$\begin{aligned} [W(\Phi)]_F + [W(\Phi^*)]_F &\equiv \int d^2\theta W(\Phi) + \int d^2\theta^\dagger W(\Phi^*) \\ &= -\frac{1}{2}W^{ij}\psi_i\psi_j + W^i F_i + \text{h.c.s}, \end{aligned} \quad (3.36)$$

as seen in eq. (3.15), which give the Yukawa interactions between  $\psi$  and  $\phi$ , holomorphic fermion mass terms, and the usual self-interaction terms for  $\phi$ . Therefore, the complete interacting Wess-Zumino Lagrangian can be written as

$$\mathcal{L}_{\text{WZ}} = [\Phi^*\Phi]_D + [W(\Phi)]_F + [W(\Phi^*)]_F. \quad (3.37)$$

To expand the model to include a gauge sector, first note that chiral superfields transform under supergauge transformations as

$$\Phi \rightarrow e^{2ig\Omega^a t^a} \Phi, \quad \Phi^* \rightarrow \Phi^* e^{-2ig\Omega^{*a} t^a}. \quad (3.38)$$

Additionally, eq. (3.32) implies that

$$e^{2g\mathcal{A}^a t^a} \rightarrow e^{2ig\Omega^{*a} t^a} e^{2g\mathcal{A}^a t^a} e^{-2ig\Omega^a t^a}. \quad (3.39)$$

Therefore, the product  $\Phi^* e^{2g\mathcal{A}^a t^a} \Phi$  is a supergauge-invariant vector superfield. Furthermore, the D-term of this expression gives the terms in eq. (3.21) as well as the gauge covariant version of eq. (3.34)

$$\begin{aligned} [\Phi^* e^{2g\mathcal{A}^a t^a} \Phi]_D &= D^\mu \phi^* D_\mu \phi + i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi - F^* F \\ &\quad - g\sqrt{2}(\phi^* t^a \psi \lambda^a + \text{h.c.}) + g\phi^* t^a \phi D^a. \end{aligned} \quad (3.40)$$

To complete the model, I need a superfield formulation for the gauge kinetic terms. One can achieve this by defining the chiral field strength superfield as

$$2g t^a \mathcal{F}_\alpha^a \equiv -\frac{1}{4} D^\dagger D^\dagger (e^{-2g\mathcal{A}^a t^a} D_\alpha e^{2g\mathcal{A}^a t^a}); \quad (3.41)$$

in the Wess-Zumino gauge, this superfield has the form

$$\mathcal{F}_\alpha^a = i\lambda_\alpha^a - \frac{i}{2}(\sigma^\mu \bar{\sigma}^\nu \theta)_\alpha A_{\mu\nu}^a + \theta^2(\sigma^\mu D_\mu \lambda^{\dagger a})_\alpha + \theta_\alpha D^a, \quad (3.42)$$



and similar for  $\mathcal{F}_a^{\dagger\dot{\alpha}}$ . Now one can see that the desired Lagrangian arises from the F-term of the square of  $\mathcal{F}$ ,

$$\frac{1}{2} [\mathcal{F}_\alpha^a \mathcal{F}_a^\alpha]_F = -\frac{1}{4} A_a^{\mu\nu} A_{\mu\nu}^a - i\lambda^{\dagger a} \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a + \frac{i}{8} A_a^{\mu\nu} \tilde{A}_{\mu\nu}^a, \quad (3.43)$$

where the final term, with  $\tilde{A}_{\mu\nu}^a \equiv \epsilon_{\mu\nu\rho\sigma} A_a^{\rho\sigma}$ , which contributes to  $CP$ -violation but is known experimentally to be highly suppressed, can be recast as a total derivative.

Finally, I can write the full Lagrangian for a gauge superfield theory:

$$\begin{aligned} \mathcal{L} = & \int d^2\theta d^2\theta^\dagger \Phi^* e^{2g\mathcal{A}^a t^a} \Phi + \int d^2\theta \left( W(\Phi) + \frac{1}{4} \mathcal{F}_\alpha^a \mathcal{F}_a^\alpha \right) \\ & + \int d^2\theta^\dagger \left( W(\Phi^*) + \frac{1}{4} \mathcal{F}_a^{\dagger\dot{\alpha}} \mathcal{F}_{\dot{\alpha}}^{\dagger a} \right), \end{aligned} \quad (3.44)$$

which describes a complete interacting theory for matter fermions and scalar and gauge bosons, as one sees in the SM, as well as the interactions of their superpartners.

## 3.2 The Minimally Supersymmetric Standard Model

In order to implement supersymmetry as part of the model of the universe, the most straightforward approach one can take is to assume that each field of the Standard Model has a superpartner with which it forms a superfield multiplet. The result of this extension is the *Minimally Supersymmetric Standard Model* (MSSM). In the MSSM, each matter fermion has a scalar superpartner called a “sfermion” (slepton, squark, stop, etc.), and each gauge boson has a fermionic gaugino partner (Wino, Bino, gluino, etc.). In each case, the SM field and its superpartner have the same quantum numbers, with the obvious exception of spin.

The Higgs scalar field also has a fermionic “Higgsino” superpartner, but some adjustments have to be made for its case, because (a) adding a single fermion to the theory with non-zero weak isospin and hypercharge would spoil gauge anomaly cancellation in the electroweak sector, and (b), as I will show in detail shortly, the requirement that the superpotential is analytic in  $\Phi$  (or  $\Phi^*$ ) forbids the simultaneous use of  $\Phi^*$  for up-type Yukawa terms and  $\Phi$  for down-type terms, as would be analogous to the SM. As a result, the MSSM must contain *two* Higgs superfields,  $H_u$  and  $H_d$ , to give mass to matter superfields of both flavors. The fields are both  $SU(2)$  doublets, with weak hypercharges  $Y_w = 1/2$  for  $H_u$  and  $Y_w = -1/2$  for  $H_d$ . The explicit forms of the doublet

superfields are

$$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}, \quad (3.45)$$

with analogous forms for the scalar bosons and Higgsino partners. As a result of this structure, the Higgs particle spectrum is significantly expanded when compared to the SM.

I will denote superfields for matter fermions as the capital letters of their SM counterparts ( $Q, U, D, L, E$ ), while I will denote the superfields of gauge bosons with their usual letters but in the calligraphic font ( $\mathcal{W}, \mathcal{B}, \mathcal{G}$ ). Superpartners for all fields will be denoted with tildes over the SM names ( $\tilde{q}, \tilde{e}, \widetilde{W}$ , etc.). This notation will stand for the remainder of the thesis. A summary of the particle content of the MSSM is given in Table 3.1.

### 3.2.1 The MSSM Lagrangian and SUSY Breaking

#### The MSSM Superpotential

The superpotential of the MSSM is highly constrained by SM gauge invariance; starting from the general form in eq. (3.35), out of all possible  $\Phi_i \Phi_j$  and  $\Phi_i \Phi_j \Phi_k$  combinations of the fields given in Table 3.1, only four terms survive. Its complete form is

$$W_{\text{MSSM}} = \epsilon_{ab} \left( -y_u^{ij} U_i^c Q_j^a H_u^b + y_d^{ij} D_i^c Q_j^a H_d^b + y_e^{ij} E_i^c L_j^a H_d^b - \mu H_u^a H_d^b \right), \quad (3.46)$$

where  $i = 1, 2, 3$  is the generation index,  $a = 1, 2$  is the  $SU(2)$  index, and color indices, which are simply contracted on the two quark fields, are not shown. The F-term of this superpotential will give rise to the following interactions:

- the SM-like mass-inducing Yukawa couplings of matter fermions  $\{u, d, e\}$  to the Higgs scalars  $h_{u,d}^0$ , of coupling strength  $y_f$  ( $f = u, d, e$ ), analogous to those seen in eq. (2.5);
- couplings of fermions (up-type to down-type) to the charged Higgs scalar fields  $h_{u,d}^\pm$ , again of strength  $y_f$ ;
- cubic scalar couplings of two sfermions  $\{\tilde{u}, \tilde{d}, \tilde{e}\}$  to a Higgs scalar of strength  $\mu^* y_f$ ;
- quartic scalar couplings of two sfermions to two Higgs scalars (*e.g.*,  $\tilde{u}\tilde{u}h_u h_u$ ) of strength  $y_f^2$ ;

Superfield	SM field	partner	$SU(3)$	$SU(2)$	$Y_w$
$Q_i$	$q_i$	$\tilde{q}_i$	<b>3</b>	<b>2</b>	1/6
$U_i$	$u_i^C$	$\tilde{u}_i^C$	<b>3</b>	<b>1</b>	2/3
$D_i$	$d_i^C$	$\tilde{d}_i^C$	<b>3</b>	<b>1</b>	-1/3
$L_i$	$\ell_i$	$\tilde{\ell}_i$	<b>1</b>	<b>2</b>	-1/2
$E_i$	$e_i^C$	$\tilde{e}_i^C$	<b>1</b>	<b>1</b>	-1
$\mathcal{B}$	$B_\mu$	$\tilde{B}$	<b>1</b>	<b>1</b>	0
$\mathcal{W}^a$	$W_\mu^a$	$\tilde{W}^a$	<b>1</b>	<b>3</b>	0
$\mathcal{G}^{a'}$	$G_\mu^{a'}$	$\tilde{G}^{a'}$	<b>8</b>	<b>1</b>	0
$H_u$	$\phi_u$	$\tilde{\phi}_u$	<b>1</b>	<b>2</b>	1/2
$H_d$	$\phi_d$	$\tilde{\phi}_d$	<b>1</b>	<b>2</b>	-1/2

Table 3.1: Superfields of the MSSM, their components, and their representations and charges under the gauge symmetries of the model.

- Higgsino-fermion-sfermion interactions (*e.g.*,  $u\tilde{u}\tilde{h}_u$ ), also of strength  $y_f$ ;
- quartic four-sfermion couplings of strength  $y_f^2$ .
- Higgs scalar mass terms for  $h_{u,d}$  with mass  $\mu^2$ ;
- Higgsino mass terms  $\mu(\tilde{h}_u^+\tilde{h}_d^- - \tilde{h}_u^0\tilde{h}_d^0) + \text{h.c.}$

There are actually a few additional terms one could add to the superpotential that are allowed by gauge invariance, but which do not conserve either *baryon number*  $B$  or *lepton number*  $L$ ; these global quantum numbers, which are automatically conserved in the SM, are assigned as  $B = \pm\frac{1}{3}$  for quarks and anti-quarks, respectively, and  $L = \pm 1$  for leptons and anti-leptons, respectively (each is zero otherwise). These values, like other quantum numbers, are present at the superfield level as well. If one were to allow terms in the superpotential which violate baryon or lepton number by one unit, *i.e.*,  $\Delta B = 1$  or  $\Delta L = 1$ , then the following terms arise:

$$W_{\Delta L=1} = \epsilon_{ab} \left( \lambda_1^{ijk} L_i^a L_j^b E_k^C + \lambda_2^{ijk} L_i^a Q_j^b D_k^C + \mu'_i L_i^a H_u^b \right) \quad (3.47)$$

$$W_{\Delta B=1} = \lambda_3^{ijk} U_i^C D_j^C D_k^C \quad (3.48)$$

We can be sure that these terms are somehow absent or extremely suppressed, because if they were present, and the couplings were  $\mathcal{O}(1)$ , tree level proton decay would arise

at ordinary energies, which is wildly inconsistent with experiment, and even with the existence of stable matter.

One way to ensure the absence of the  $B$ - and  $L$ -violating terms is to enforce the discrete symmetry  $R$ -parity, which is defined as

$$R = (-1)^{3(B-L)+2s},$$

where  $s$  is spin. One can determine that all SM matter fermions and Higgs bosons have  $R = 1$ , while all SUSY particles have  $R = -1$ . Enforcement of  $R$ -parity means every interaction vertex has  $R = 1$  overall, which has several important implications: (a) any vertex will contain an even number of SUSY fields, and SUSY particles will always be produced in even numbers, (b) the product of any SUSY particle decay will contain an odd number of new SUSY fields, and (c) the lightest SUSY particle (LSP) is stable and will be present at the end of any SUSY decay process. The stability of the LSP, if taken with the cosmologically-motivated requirement that it be electrically and color neutral, suggests that it is an excellent candidate for the composition of non-baryonic dark matter.

While  $R$ -parity may seem ad-hoc despite empirical motivations for its existence, it actually has theoretical motivation as well in the context of grand unified theory and some  $SO(10)$  models, in particular, due to its relationship to  $B - L$  symmetry, which is typically gauged at high energies in  $SO(10)$  and is central to the seesaw mechanism for neutrino masses. I will discuss these topics further in the next chapter.

## Soft SUSY Breaking in the MSSM

The soft SUSY breaking terms of the MSSM are those of the forms in eq. (3.24) that are consistent with gauge invariance and  $R$ -parity. They are

$$\begin{aligned} \mathcal{L}_{\text{soft}} = & -\frac{1}{2} \left( M_1 \tilde{B} \tilde{B} + M_2 \tilde{W}^a \tilde{W}^a + M_3 \tilde{G}^{a'} \tilde{G}^{a'} + \text{h.c.} \right) \\ & + \epsilon_{ab} \left( -a_u^{ij} \tilde{u}_i^c \tilde{q}_j^a H_u^b + a_d^{ij} \tilde{d}_i^c \tilde{q}_j^a H_d^b + a_e^{ij} \tilde{e}_i^c \tilde{\ell}_j^a H_d^b + \text{h.c.} \right) \\ & - (m_{\tilde{q}}^2)^{ij} \tilde{q}_i^\dagger \tilde{q}_j - (m_{\tilde{\ell}}^2)^{ij} \tilde{\ell}_i^\dagger \tilde{\ell}_j - (m_{\tilde{u}}^2)^{ij} \tilde{u}_i^c \tilde{u}_j^{c*} - (m_{\tilde{d}}^2)^{ij} \tilde{d}_i^c \tilde{d}_j^{c*} - (m_{\tilde{e}}^2)^{ij} \tilde{e}_i^c \tilde{e}_j^{c*} \\ & - m_{h_u}^2 h_u^\dagger h_u - m_{h_d}^2 h_d^\dagger h_d - b \epsilon_{ab} (h_u^{*a} h_d^b + \text{h.c.}); \end{aligned} \quad (3.49)$$

the summation over  $a, a'$  for the gauginos runs over the generators, while the  $\epsilon$  contraction in the  $a$ -terms and  $b$  Higgs term is over  $SU(2)$  indices as it was in (3.46). The daggers on the scalars in the mass squared terms indicate complex conjugate of the scalar but transpose in  $SU(2)$  space. Note that unlike the Yukawa couplings  $y_f$ , the  $a_f$  couplings have mass dimension. Since all the fields here acquire masses after EWSB from the couplings in  $W_{\text{MSSM}}$ , one expects physical masses to be generated by a mixing of all relevant terms.

The soft breaking terms introduce 105 new parameters to the theory, including numerous mixing angles and phases in addition to the masses themselves. This fact is quite disconcerting without further context; however, several important experimental considerations lead to substantial constraints on the full parameter space. For instance, the absence of evidence for substantial  $CP$  violation in the universe requires that phases are small or zero. Both the  $a_e$  and  $m_e^2$  terms contribute to *lepton flavor violation* (LFV), which is the breaking of global lepton flavor number symmetries present in the SM; this phenomenon occurs in processes such as  $\mu \rightarrow e\gamma$  and must be at least highly suppressed to agree with experimental limits [31]. The presence of arbitrary mass matrices  $m_{\tilde{f}}^2$  would also disrupt the suppression of *flavor changing neutral currents* (FCNC), which are exactly zero at tree level in the SM and suppressed even at loop level through cancellation. Experimental limits on processes such as  $K^0 \rightarrow \bar{K}^0$ , *i.e.*,  $d\bar{s} \rightarrow s\bar{d}$ , strongly constrain the squark mass differences [32].

These considerations motivate an extreme simplification of the soft breaking parameter space, built on the following assumptions:

$$a_f^{ij} \simeq A_f y_f^{ij}; \quad (m_{\tilde{f}}^2)^{ij} \simeq m_{\tilde{f}}^2 \delta^{ij}; \quad \text{Im}\{A_f, M_i\} \simeq 0; \quad (3.50)$$

These simplifications are the SUSY-scale realization of a high-energy prescription known as *universality*, which I will discuss in more detail below.

There are several feasible mechanisms for dynamically generating the soft breaking terms; each involves a *hidden sector*, which couples very weakly or not at all to the “visible” sector of SM superpartners, and a *messenger sector*, which *mediates* the hidden sector physics, *i.e.* “relays” it to the visible sector, creating the soft terms seen in (3.49). Popular mechanisms for SUSY breaking are *gravity-mediated breaking*, in which a hidden sector auxiliary vev  $\langle F \rangle$  is communicated to the MSSM fields through gravitational effects, and *gauge-mediated breaking*, in which a similar vev is coupled to messenger fields charged under the SM gauge group, so that soft terms arise through

multi-loop order interactions between the messenger fields and MSSM fields via the SM bosons. Since the gauge bosons are blind to generation and, in some cases, flavor in general, the conditions in (3.50) may be naturally present. Other possible mediators include anomalies and extra-dimensions. There is little agreement on which mediator is “most” appropriate or promising, as every prescription faces a list of at least minor phenomenological issues.

Gravity and gauge mediation can also be readily explored in *supergravity*, which arises automatically when one considers local supersymmetry transformations, *i.e.*, gauged supersymmetry. The gauging of supersymmetry unifies global SUSY with the spin-2 field theory of the graviton. In this theory, the fermionic Goldstone mode associated with the broken SUSY generator is eaten by the spin-3/2 graviton superpartner, the *gravitino*. Depending on the mediator, the gravitino may have cosmological or even TeV scale consequences. Additionally, an appropriately “minimal” supergravity model gives rise to flavor *universality*, mentioned above, where at the GUT scale  $M_U$ ,

$$\begin{aligned} A_u = A_d = A_e \equiv A_0, \quad m_{\tilde{f}}^2 = m_{h_u}^2 = m_{h_d}^2 \equiv m_0^2 \quad \forall f, \\ b = B_0\mu, \quad M_1 = M_2 = M_3 \equiv m_{1/2}, \end{aligned} \tag{3.51}$$

where the parameters  $A_0, B_0, m_0, m_{1/2}$  are all determined by the theory in terms of  $\langle F \rangle$  and  $M_{\text{Pl}}$ . The weaker conditions seen in (3.50) arise through the running of the parameters down from  $M_U$  to the soft breaking scale  $M_{\text{SUSY}}$ . As I will discuss shortly, taking universality at the GUT scale means that it coincides with unification of the standard model gauge couplings  $g_s, g, g'$  in the MSSM, which will be a key factor in motivating the synthesis of SUSY with  $SO(10)$  grand unification. I will assume universality throughout the remainder of this work.

## The Complete MSSM Lagrangian and EWSB

With  $W_{\text{MSSM}}$  and  $\mathcal{L}_{\text{soft}}$  defined, I can write the complete MSSM Lagrangian, in terms of superfields, as

$$\begin{aligned}
\mathcal{L}_{\text{MSSM}} = & \int d^2\theta d^2\theta^\dagger \left\{ Q_i^* \exp \left( 2g_s \mathcal{G}^{a'} \lambda^{a'} + 2g \mathcal{W}^a T^a + g' \mathcal{B}/3 \right) Q_i + \right. \\
& U_i^{\mathcal{C}*} \exp \left( 2g_s \mathcal{G}^{a'} \lambda^{a'} + 4g' \mathcal{B}/3 \right) U_i^{\mathcal{C}} + D_i^{\mathcal{C}*} \exp \left( 2g_s \mathcal{G}^{a'} \lambda^{a'} - 2g' \mathcal{B}/3 \right) D_i^{\mathcal{C}} + \\
& L_i^* \exp \left( 2g \mathcal{W}^a T^a - g' \mathcal{B} \right) L_i + E_i^{\mathcal{C}*} \exp \left( -2g' \mathcal{B} \right) E_i^{\mathcal{C}} \\
& \left. + H_u^* \exp \left( 2g \mathcal{W}^a T^a + g' \mathcal{B} \right) H_u + H_d^* \exp \left( 2g \mathcal{W}^a T^a - g' \mathcal{B} \right) H_d \right\} \\
& + \int d^2\theta \left( W_{\text{MSSM}} + \frac{1}{4} \mathcal{G}_\alpha^{a'} \mathcal{G}_\alpha^{a'} + \frac{1}{4} \mathcal{W}_\alpha^a \mathcal{W}_\alpha^a + \frac{1}{4} \mathcal{B}_\alpha \mathcal{B}_\alpha \right) + \text{c.c.} + \mathcal{L}_{\text{soft}}. \quad (3.52)
\end{aligned}$$

The D-terms for the chiral superfields in this Lagrangian will give rise to the following interactions:

- the SM kinetic terms and gauge boson interactions for the fermions  $\{u, d, e\}$  and Higgs bosons  $\{h_u, h_d\}$ ;
- the kinetic terms and gauge boson interactions of the SM superpartners  $\{\tilde{u}, \tilde{d}, \tilde{e}, \tilde{h}_u, \tilde{h}_d\}$ , which include cubic sfermion-sfermion-boson terms (*e.g.*,  $\tilde{f}\tilde{f}W$ ) of coupling strength  $g$ , quartic terms involving two sfermions and two gauge bosons (*e.g.*,  $\tilde{f}\tilde{f}WW$ ) of strength  $g^2$ , and cubic higgsino-higgsino-boson terms (*e.g.*,  $\tilde{h}\tilde{h}W$ ) of strength  $g$ ;
- cubic fermion-sfermion-gaugino (*e.g.*,  $f\tilde{f}\widetilde{W}$ ) terms of coupling strength  $g$ ;
- quartic four-sfermion and four-Higgs boson terms of strength  $g^2$ .

The F-terms of the gauge field strength terms in this Lagrangian will give rise to the following interactions:

- the SM kinetic terms and self-interaction terms for the gauge bosons  $\{G^a, W^a, B\}$ ;
- the kinetic terms for the gaugino superpartners  $\{\widetilde{G}^a, \widetilde{W}^a, \widetilde{B}\}$  and their cubic gaugino-gaugino-boson self interactions of strength  $g$ .

The neutral Higgs scalar potential for the model is

$$V_h = (|\mu|^2 + m_{h_u}^2)|h_u^0|^2 + (|\mu|^2 + m_{h_d}^2)|h_d^0|^2 - (B_0\mu h_u^0 h_d^0 + \text{c.c.}) + \frac{1}{8}(g^2 + g'^2)(|h_u^0|^2 - |h_d^0|^2)^2, \quad (3.53)$$

where I've set  $h_u^+ = h_d^- = 0$  at the minimum (without loss of generality) to avoid disturbing electromagnetism. Both  $h_u^0$  and  $h_d^0$  acquire vevs to break EW symmetry. The values of  $B_0$ ,  $\langle h_u^0 \rangle$ , and  $\langle h_d^0 \rangle$  can all be chosen and real and positive through field redefinition and  $U(1)_Y$  gauge transformation. I'll define  $\langle h_u^0 \rangle \equiv v_u$  and  $\langle h_d^0 \rangle \equiv v_d$ ; the two vevs relate to the SM vev as  $v_u^2 + v_d^2 = v^2$ , where  $v = 174 \text{ GeV}$  (or  $246 \text{ GeV}/\sqrt{2}$ , as an alternate convention). It's customary to define

$$\tan \beta = \frac{v_u}{v_d}, \quad v_u < v_d,$$

so that  $v_u = v \sin \beta$  and  $v_d = v \cos \beta$ .

Of the eight real scalar degrees of freedom in the two complex Higgs doublets, three become the Goldstone bosons, eaten by the massive gauge bosons after EWSB, which leaves five physical Higgs scalars in the model. There are two charged bosons  $h^\pm$ , two neutral,  $CP$ -even bosons  $h^0$  and  $H^0$ , and one neutral,  $CP$ -odd pseudo-scalar  $A$ ; the lighter of the neutral scalars corresponds to the Higgs of the standard model. The tree-level masses of the neutral bosons can be written as

$$m_{H,h}^2 = \frac{1}{2} \left\{ m_A^2 + M_Z^2 \pm \sqrt{(m_A^2 + M_Z^2)^2 - 4m_A^2 M_Z^2 \cos^2 2\beta} \right\}, \quad (3.54)$$

where

$$m_A^2 = 2|\mu|^2 + m_{h_u}^2 + m_{h_d}^2.$$

One might notice that the lighter SM scalar mass is less than  $M_Z$ , at least at tree level and for  $m_A > M_Z$ . If one includes the largest loop correction, coming from the top and stop couplings, one can obtain  $m_h$  of up to about 135 GeV or, which puts the observed Higgs mass near the upper end of the comfortably consistent parameter space of the MSSM.

In a manner similar to the mixing of the gauge bosons seen in the SM, there is additional mixing among like-charged superpartners in the MSSM. In particular, the like-charged Winos  $\widetilde{W}^\pm$  and Higgsinos  $\tilde{h}_{u,d}^\pm$  mix to give the physical *charginos*  $\chi^\pm$ , and



the two neutral gauginos  $\widetilde{B}, \widetilde{W}^0$  and Higgsinos  $\widetilde{h}_{u,d}^0$  mix to give the four *neutralinos*  $\chi_i^0$ . Since  $SU(3)_C$  is unbroken in the model, the *gluinos*  $\widetilde{g}$ , which would be massless in the absence SUSY breaking, degenerately share the soft-breaking Majorana mass  $M_3$ .

The particle and anti-particle fermion superpartners will also generally mix with one another. The two physical scalar partners are typically denoted simply by  $\widetilde{f}_{1,2}$ .

### 3.2.2 Gauge Coupling Unification

In addition to solving the hierarchy problem, one of the more curious and inviting features of the MSSM is the rather precise unification of the three SM gauge couplings at high energies. To understand the meaning of this statement, recall that, as mentioned briefly in the previous chapter, the physical parameters of a gauge field theory actually change with the energy scale of interaction due to renormalization effects. The evolution of a gauge coupling  $g$  is governed by the *beta function* [33]

$$M \frac{\partial g}{\partial M} = \beta(g), \quad (3.55)$$

where  $M$  is the energy scale in question, referred to as simply the renormalization scale. The derivative here is often seen written as  $\partial/\partial(\ln M)$  or  $\partial/\partial t$ , with  $t \equiv \ln M$ , for simplicity. Taking the above expression as an equation of evolution, one can see that the running with energy of  $g$  is a function of  $g$  itself; furthermore,  $\beta(g)$  will be a smooth function such that the evolution can be viewed as a continuous, group-like transformation for  $M \rightarrow M + \delta M$ . As a result, eq. (3.55) is known as the *renormalization group equation* (RGE) for  $g$ . For a general gauge theory, the beta function due to single-loop-level corrections is

$$\beta(g) = \frac{bg^3}{16\pi^2} \equiv \frac{g^3}{16\pi^2} \left( -\frac{11}{3}C_2(G) + \frac{4}{3}n_f C(r) \right), \quad (3.56)$$

where  $n_f$  is the number of fermions charged under the group in the theory, and  $C_2(G)$  and  $C(r)$  are group theory factors. For an  $SU(N)$  theory,  $C_2(G) = N$ , while  $C_2(G) = 0$  for an abelian group; In the SM,  $C(r)$  is normalized to  $1/2$  for  $SU(2)_L$  and  $SU(3)_C$  and to  $3Y^2/5$  for  $U(1)_Y$ . This unusual normalization for  $U(1)_Y$  is chosen to match the redefinition of the gauge coupling  $g'$  used in  $SU(5)$  and  $SO(10)$  grand unification, which I will discuss in more detail in the next chapter. For a semi-simple theory of multiple gauge groups such as the SM, one can consider a separate, independent RGE for each

coupling in the theory:

$$M \frac{\partial g_i}{\partial M} = \frac{b_i g_i^3}{16\pi^2}, \quad (3.57)$$

for multiple couplings  $g_i$ . Notice that, given the beta function for an  $SU(N)$  coupling, the beta function will be negative for sufficiently small  $n_f$ , which implies that the strength of the coupling diminishes with increasing energy. As a result, the coupling strength should vanish at some high energy. This property, known as *asymptotic freedom*, is a feature of both non-Abelian symmetries of the SM.

For the standard model, careful counting of fields reveals that

$$b_i = \left( \frac{41}{10}, -\frac{19}{6}, -7 \right), \quad (3.58)$$

where I've made the identifications  $g_3 = g_s$ ,  $g_2 = g$ , and  $g_1 = \sqrt{\frac{5}{3}}g'$ ; Again, the change in normalization for  $g'$  is made for compatibility with  $SU(5)$  grand unification. Conveniently, if one writes the RGEs above in terms of the parameters  $\alpha_i = g_i^2/4\pi$ , the resulting equations (still at one-loop order) are linear in  $\alpha_i^{-1}$ :

$$M \frac{\partial \alpha_i^{-1}}{\partial M} = \frac{b_i}{2\pi}. \quad (3.59)$$

As a result, the running of the couplings will be straight lines on a plot of coupling strength vs.  $\log M$ . That plot is given for the three SM couplings in Figure 3.2, shown as the black dashed lines in the plot. Perhaps unexpectedly, the values of the three couplings show signs of attempting to merge in the vicinity of  $10^{13}$  GeV; this is a very tantalizing concept... could it be that at very high energies, and hence in the very early universe, the strong and electroweak forces were just different components of a single interaction? This is of course similar to what we see in electroweak unification; before EWSB, massless  $W^a$  and  $B$  bosons would have mediated a single and perhaps long-range electroweak force, resulting in a presumably unrecognizable universe. In the end, it seems reasonable or even wise to assume that the merging of forces continues as one moves back in time, and up in energy, toward the big bang.

Yet, this vague trend in the SM is only the beginning of the story. In the MSSM, due to the additional fields of varying species, the beta function becomes

$$\beta_S(g) = \frac{g^3}{16\pi^2} \left( -3C_2(G) + \sum_{\phi} C(r(\phi)) \right), \quad (3.60)$$

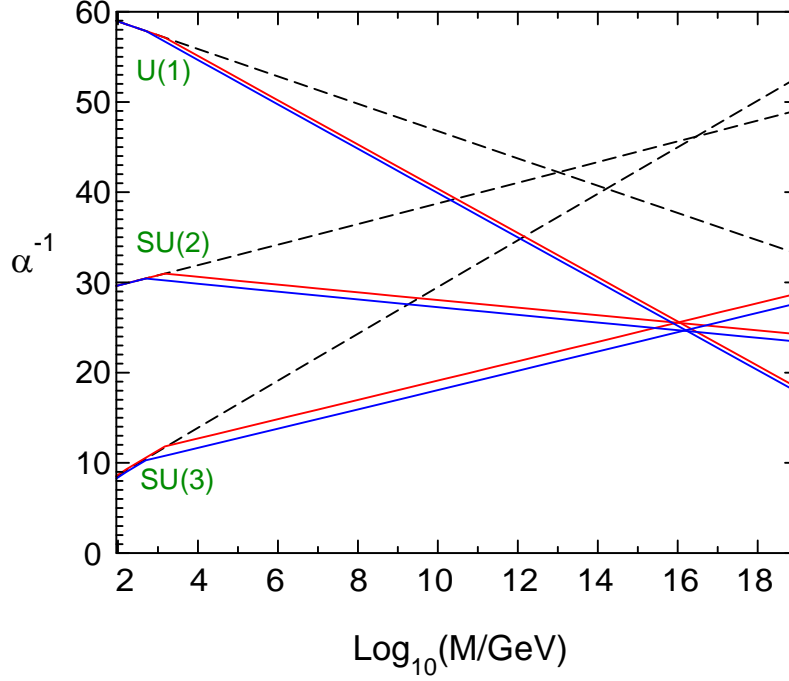


Figure 3.2: Renormalization group evolution of the inverse gauge couplings  $\alpha_i^{-1}$  for the SM (dashed lines) and the MSSM (solid colored lines) [28]; for the MSSM case, the red vs. blue colored lines give bounds under variation of the superpartner masses.

where the sum over fields  $\phi$  includes all the matter and Higgs fields in the theory and their superpartners. The values of the coefficients are

$$b_i^S = \left( \frac{33}{5}, 1, -3 \right). \quad (3.61)$$

Note that the beta function for  $SU(2)_L$  has changed signs. Looking again at Figure 3.2, the solid colored lines show the running of  $\alpha_i^{-1}$  in the MSSM; the red and blue lines for each coupling give variation for a range of superpartner masses 0.5-1.5 TeV. The merging of the coupling strengths has improved dramatically, with a nearly exact agreement between the three coupling values at an energy scale of  $\sim 2 \times 10^{16}$  GeV. This behavior, known as *gauge coupling unification*, seems almost too good to be true, but does in fact arise for reasonable or even preferred values for the parameters of the theory. Now perhaps one can see why the prospect of combining theories of SUSY with those of grand unification became so popular: this feature of the MSSM compels us to explore the possibility that this merger is no accident. Adding unification to the

hierarchy problem solution and prospects for dark matter, the lucrative nature of the MSSM is clear, and one might understand why it created so much excitement for BSM physics, and why its presence in BSM theories persists to this today, even despite an increasingly long list of phenomenological difficulties.

Note though that I have still made no further mention of neutrino masses, which, again, are strongly suggested by empirical data. Adding neutrino masses to the MSSM is quite analogous to adding them in the standard model, although the allowed soft breaking terms contribute further to lepton flavor violation and the other phenomenological complications discussed previously in the context of the charged fermions. Even if one avoids those issues as before, it remains that extending the MSSM to accommodate neutrino mass phenomenology is starkly *ad hoc*. In the context of grand unification, however, this is not the case. A rather attractive mechanism for describing neutrino masses goes hand-in-hand with  $SO(10)$  grand unification, which will be the topic of the next chapter.

## Chapter 4

# Grand Unification and Neutrino Mass

Once the theory of electroweak unification and its spontaneous breakdown via the Higgs mechanism were fully understood, grand unification was perhaps an easy target for physicists looking to go beyond the standard model. If the acquisition of a vev by a scalar boson could break  $SU(2)_L \times U(1)_Y$  down to  $U(1)_{\text{em}}$  and a short-range weak force via massive vector bosons, then perhaps there could be more such scalars, of even larger mass, governing additional spontaneous breakdowns of higher dimensional groups to  $SU(3)_C \times SU(2)_L \times U(1)_Y$ . Such a breakdown process would correspond to the physical notion that the original symmetry of our universe was quite a simple one (which can be taken literally in the context of group theory), forced into a more elaborate configuration by the nontrivial internal landscape of the quantum vacuum as spacetime expanded and average energy density fell.

Yet, as previously mentioned, there are many reasons beyond aesthetics to pursue unification. In addition to the highly suggestive nature of gauge coupling unification discussed at the end of the previous chapter, GUT models explain the seemingly arbitrary values for hypercharge in the SM and consequently offer some basis for charge quantization; they often restore parity symmetry in the gauge group; and they may provide a framework more conducive to giving neutrinos mass. Furthermore, specifically in the case of  $SO(10)$ , the right-handed neutrino appears automatically, and neutrino masses arise quite naturally, in connection to unification-scale breaking of  $B - L$ .

## 4.1 Earlier Models of Unification

### 4.1.1 Pati-Salam and Left-Right Symmetry

J.C. Pati and A. Salam proposed the first model of partial unification in 1974 [4], based on the gauge group  $SU(2)_L \times SU(2)_R \times SU(4)_C$ . The model treated lepton number as the fourth color, and the resulting multiplets predictably contained new fields with “lepto-quark” characteristics.

Left-right symmetric models restore the maximal breaking of parity seen in the SM gauge group. These models were first developed by R.N. Mohapatra, G. Senjanovic, and Pati [34], also during 1974.<sup>1</sup> The simplest  $L$ - $R$  model is based on the gauge group  $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ , where the couplings are  $g_{2L} = g_{2R}$  and  $g'$ . Such models are really extensions of the SM model rather than unification models, since no SM model multiplets are merged into larger representations. With the addition of the  $SU(2)_R$  gauge group and the presence of  $U(1)_{B-L}$ , one can define electric charge as [36]

$$Q = T_L^3 + T_R^3 + \frac{1}{2}(B - L);$$

this definition provides explanations for not only the seemingly arbitrary values for hypercharge seen in the SM, but also for the quantization of electric charge.

Since  $SU(4) \supseteq SU(3) \times U(1)$ , the left-right model can be naturally embedded into Pati-Salam.

Left-right symmetry adds right-handed  $W$  and  $Z$  bosons to the SM and collects the  $SU(2)_L$ -singlet fermions into doublets of their own:

$$q_R \equiv \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \quad \ell_R \equiv \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}; \quad (4.1)$$

here, finally, one sees the addition of the right handed neutrino to the model. Since right-handed neutrinos are not observed in our low-energy world, the model will need some way to understand this. The most popular solution utilizes the Majorana character of neutrinos as follows. Consider the following scalar fields with  $SU(2)_L \times SU(2)_R \times$

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<sup>1</sup>Right-handed currents had first been proposed in the context of the SM by Mohapatra in 1972, as a possible source of  $CP$  violation [35].

$U(1)_{B-L}$  representations [9]:

$$\Delta_L : (\mathbf{3}, \mathbf{1}, 2), \quad \Delta_R : (\mathbf{1}, \mathbf{3}, 2), \quad \phi : (\mathbf{2}, \mathbf{2}, 0).$$

I can write interactions between these Higgs fields and the leptons (for one generation) as

$$\mathcal{L}_{\text{Yuk}} \ni h \bar{\ell}_L \phi \ell_R + \tilde{h} \bar{\ell}_L \tilde{\phi} \ell_R + i f \left( \ell_L^T C^{-1} \sigma_2 \sigma_a \Delta_L^a \ell_L + \ell_R^T C^{-1} \sigma_2 \sigma_a \Delta_R^a \ell_R \right) + \text{h.c.s}, \quad (4.2)$$

where  $\psi^T C^{-1} \psi$  is the Lorentz scalar for Majorana fermions, and where  $\tilde{\phi} = \sigma_2 \phi^* \sigma_2$ . The chiral Majorana interactions here violate lepton number conservation by 2 units but conserve  $B - L$ . The  $SU(2)$  structure of these terms couples the neutrino to the neutral component of  $\Delta$  for both the left and right cases; hence, if either field acquires a vev, the neutrinos will receive Majorana contributions to their masses. A vev for  $\phi$  will play the role of breaking EWSB and giving masses to all of the fermions, including contributions to the neutrinos. However, if  $\langle \Delta_R \rangle \gg \langle \phi \rangle, \langle \Delta_L \rangle$ , then the right handed neutrinos will acquire masses much heavier than the rest of the fields, which would explain their absence in nature. The vev  $\langle \Delta_R \rangle$  will also serve to break  $SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y$  if parity is broken in conjunction.

A closer look at the full neutrino mass matrix will reveal that the left-handed neutrinos are  $m_\nu \sim \langle \phi \rangle^2 / \langle \Delta_R \rangle$ , and are thus suppressed by the heavy scale. Furthermore, if the vev  $\langle \phi \rangle$  is inversely hierarchical, then the solutions to the scalar potential give  $\langle \Delta_L \rangle \sim 0$ , resulting in extremely small masses for the left-handed neutrinos, also in agreement with observation. This prescription, known as the *seesaw mechanism*, has held as the most phenomenologically viable explanation for neutrino mass for 35 years. It is also quite compatible with  $SO(10)$  unification. I will discuss the mechanism in more detail shortly.

### 4.1.2 $SU(5)$ Grand Unified Theory

Georgi and Glashow introduced the first model of complete grand unification [3] in the same year as Pati-Salam, based on the gauge group  $SU(5)$ . The SM gauge group has rank  $r = 4$ , where the rank of a Lie group is given by the dimension of its *maximal Cartan sub-algebra*, i.e., by the number of diagonal generators in the algebra. A group

can only be embedded in a larger group if  $r_{\text{small}} \leq r_{\text{large}}$ , and  $SU(5)$  is the smallest simple Lie group of rank-4; therefore, it is the smallest simple group in which the SM group can be embedded, and  $SU(3) \times SU(2) \times U(1)$  is a maximal subgroup.

The 15 matter fields per generation in the SM can be embedded into  $SU(5)$  using the conjugate fundamental representation  $\bar{\mathbf{5}} \ni \{\ell, d_\rho^c\}$  and the completely antisymmetric two-index representation  $\mathbf{10} \ni \{q_\rho, u_\rho^c, e^c\}$ ;  $\rho = 1, 2, 3$  is the color index. Their explicit forms are

$$\psi_a \equiv \begin{pmatrix} d_{\bar{r}}^c \\ d_{\bar{g}}^c \\ d_{\bar{b}}^c \\ e \\ \nu \end{pmatrix}_L ; \quad \chi_{ab} \equiv \begin{pmatrix} 0 & u_b^c & -u_{\bar{g}}^c & u_r & d_r \\ & 0 & u_{\bar{r}}^c & u_g & d_g \\ & & 0 & u_b & d_b \\ & & & 0 & e^c \\ & & & & 0 \end{pmatrix}_L . \quad (4.3)$$

The model has 24 generators, and thus 24 gauge bosons, which decompose under the SM group as

$$\{24\} = G(\mathbf{8}, \mathbf{1}, 0) \oplus W(\mathbf{1}, \mathbf{3}, 0) \oplus Y_w(\mathbf{1}, \mathbf{1}, 0) \oplus X_\rho^{u,d}(\mathbf{3}, \mathbf{2}, -\frac{5}{6}) \oplus \bar{X}_{\bar{\rho}}^{u,d}(\bar{\mathbf{3}}, \mathbf{2}, \frac{5}{6}),$$

where the first three components correspond to the gluons,  $W$  bosons, and hypercharge boson, respectively. The remaining two components carry both color and weak isospin; these fields are understood as 12 new individual  $SU(5)$  bosons, which allow quark-lepton interaction at a single vertex. The coupling  $g_5$  to all bosons is universal, as  $g_5 = g_3 = g_2 = g_1 = \sqrt{\frac{5}{3}}g'$  at the unification scale  $M_U$ .

Note that in order to write the diagonal hypercharge generator such that it preserves  $SU(3)_C$ , one will find that the diagonal entries are fully determined by a single parameter plus the overall normalization, and hence the action of this generator on the various component fields fixes the values of  $Y_w$  for all the SM fermions precisely as needed. Quantization of electric charge follows as an implication.

The Higgs sector of  $SU(5)$  has a minimum content of a 24-dimensional adjoint field  $\Phi$  and a 5-dimensional fundamental field  $H_5$ . Breaking  $SU(5) \rightarrow G_{\text{SM}}$  occurs via a vev  $\langle \Phi \rangle_{24}$ , aligned with the diagonal ( $\sim$  hypercharge) generator  $\lambda_{24}$ . The breaking gives masses to the  $X$  bosons  $M_X^2 \sim g_5^2 V^2$ , where  $\langle \Phi \rangle = V \lambda_{24}$ .

The  $\mathbf{5}$  Higgs is essentially  $(H_C^p \oplus \phi_{\text{SM}})$ , *i.e.*, a color triplet Higgs field and the SM Higgs doublet in a single multiplet. EWSB occurs through the vev  $\langle H_5 \rangle = (0, 0, 0, 0, v)^T$ ,



which gives mass to the fermions through the couplings

$$\mathcal{L}_{\text{Yuk}} = h_{ij} \bar{\psi}_a^i \chi_{ab}^j H_b^\dagger + h'_{ij} \epsilon^{abcde} \chi_{ab}^{Ti} C^{-1} \chi_{cd}^j H_e + \text{h.c.s.} \quad (4.4)$$

The down-type and charged lepton masses are both given by the first Yukawa term in the expression; as a result  $m_e^i = m_d^i$  for  $i = 1, 2, 3$ . While these relationships are given at the unification scale, only third generation Yukawa runnings are substantial enough to correct the experimental inaccuracy of this relationship at low energies (because  $m_b \sim m_\tau$ ). In order to give realistic mass eigenvalues to all the down-type fields, one can introduce a **45**-dimensional Higgs field  $H_{ab}^c$ .

Expansion of the  $X$  gauge boson couplings to the matter multiplets gives interactions with the individual fields of the form

$$\begin{aligned} \mathcal{L}_X = & -\frac{g_5}{\sqrt{2}} X_\mu^{u\rho} (\epsilon_{\rho\sigma\tau} \bar{u}_L^{C\sigma} \gamma^\mu u_L^\tau + \bar{d}_{L\rho} \gamma^\mu e_L^C + \bar{d}_{R\rho} \gamma^\mu e_R^C) \\ & -\frac{g_5}{\sqrt{2}} X_\mu^{d\rho} (\epsilon_{\rho\sigma\tau} \bar{u}_L^{C\sigma} \gamma^\mu d_L^\tau - \bar{u}_{L\rho} \gamma^\mu e_L^C + \bar{d}_{R\rho} \gamma^\mu \nu_R^C) + \text{h.c.s.} \end{aligned} \quad (4.5)$$

Note that some vertices include quark-lepton mixing. As a result, through the exchange of an  $\bar{X}^u$  boson, the process

$$uu \rightarrow de^+$$

is possible. Similarly,

$$ud \rightarrow ue^+$$

can occur through the exchange of a  $\bar{X}^d$ . Either process may therefore lead to the decay of a nucleon. In particular, one sees

$$\tau(p \rightarrow \pi^0 e^+) \approx \frac{M_X^4}{g_5^4 m_p^5}$$

When  $SU(5)$  theory was new, limits on proton lifetime were in the vicinity of  $10^{28-30}$  GeV [37], which implied  $M_X \gtrsim 10^{14-15}$  GeV. Since then, lifetime limits have risen by several orders of magnitude, and consequently the basic  $SU(5)$  model has been virtually ruled out as a viable theory of nature (a few niches in the parameter space do technically remain). One can make extensions to the model to salvage its validity, although most require severe tuning of free parameters.

Other shortcomings of the model exist as well. Like the SM, the  $SU(5)$  model

suffers from a “gauge hierarchy problem”, in that there is no basis for the extreme difference of the EW and unification scales. Additionally, as in the SM and the MSSM, extension of the model to include neutrino mass is completely *ad hoc*. However, the  $SU(5)$  model can be embedded into the larger group  $SO(10)$ , in which neutrino masses arise naturally. In fact, specifically in the SUSY case, all of the above concerns see at least partial resolution.

Before discussing  $SO(10)$  models, I will discuss the seesaw mechanism for neutrino mass in more detail.

## 4.2 The Seesaw Mechanism and Neutrino Masses

Looking back at section 4.1.1, one can take the form of the Higgs fields in the left-right model as [9]

$$\Delta_{L,R} \equiv \sigma^a \Delta_{L,R}^a = \begin{pmatrix} \delta^+/\sqrt{2} & \delta^{++} \\ \delta^0 & -\delta^+/\sqrt{2} \end{pmatrix}_{L,R}, \quad \phi \equiv \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix};$$

if the neutral components of the fields acquire vevs, I can write them without loss of generality as

$$\langle \Delta_{L,R} \rangle = \begin{pmatrix} 0 & 0 \\ v_{L,R} & 0 \end{pmatrix}, \quad \langle \phi \rangle = e^{i\alpha} \begin{pmatrix} \kappa & 0 \\ 0 & \kappa' \end{pmatrix}. \quad (4.6)$$

Now if I expand eq. (4.2) into components of the  $SU(2)_{L,R}$  multiplets, one finds the following neutrino mass terms:

$$\mathcal{L}_{\text{Yuk}} \ni h_\nu \bar{\nu}_L \nu_R (\kappa + \kappa') e^{i\alpha} + f v_L \nu_L^T C^{-1} \nu_L + f v_R \nu_R^T C^{-1} \nu_R + \text{h.c.s.} \quad (4.7)$$

Looking at the resulting neutrino mass matrix, in terms of its Weyl components, one sees that, neglecting the phase  $\alpha$ ,

$$\mathcal{M}_\nu = \begin{pmatrix} f v_L & \frac{1}{2}(h\kappa + \tilde{h}\kappa') \\ \frac{1}{2}(h\kappa + \tilde{h}\kappa') & f v_R \end{pmatrix}; \quad (4.8)$$

The scalar potential for  $\Delta_{L,R}$  and  $\phi$  is quite extensive, but under the assumption that  $\kappa' \ll \kappa$  as well as  $\kappa \ll v_R$ , one finds that

$$v_L \simeq \frac{r\kappa^2}{2v_R} \ll 1, \quad (4.9)$$

where  $r$  is a combination of parameters from the potential and is generally small. Hence, the vev  $v_L$  will be highly suppressed, and one finds the following eigenvalues for  $\mathcal{M}_\nu$ :

$$m_\nu \simeq f v_L - \frac{h^2 \kappa^2}{2f v_R}, \quad M_N \simeq 2f v_R, \quad (4.10)$$

where  $N$  is the heavy  $\sim$ right-handed neutrino; the mass eigenstates are generally linear combinations of  $\nu_{L,R}$ , but the extremely hierarchical nature of the mass matrix leads to large suppression of the mixing for the single-generation case.

This “seesaw” mechanism can be explored outside of the context of left-right symmetry as well. In fact, one may consider simply adding the right-handed neutrino to the SM under the assumptions that it must be *sterile*, *i.e.*, a singlet under the full gauge group, and that it is Majorana and heavy. Then the model is extended through the inclusion of the terms

$$\mathcal{L}_{SM} \ni y_\nu^{ij} \epsilon_{\alpha\beta} \bar{\ell}_{Li}^\alpha \phi^{*\beta} \nu_{Rj} + \frac{1}{2} M_N^i \nu_{Ri}^T C^{-1} \nu_{Ri} + \text{h.c.s}; \quad (4.11)$$

after EWSB, one finds a neutrino mass matrix similar in form to (4.8):

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & y_\nu^{ij} v \\ y_\nu^{ji} v & \delta_{ij} M_N^j \end{pmatrix}, \quad (4.12)$$

which will give left-handed eigenvalues of the form

$$m_\nu \simeq -\frac{y_\nu^2 v^2}{M_N}. \quad (4.13)$$

This form for neutrino mass, involving a Majorana term for the heavy right-handed neutrinos only, is known as the *type-I seesaw*. Integrating out the heavy neutrinos leads to an effective dimension-5 operator of the form

$$\mathcal{L}_{SM,\text{eff}} \simeq \frac{y_\nu^2}{M_N} \bar{\ell}_\alpha \phi^\alpha \ell^\beta \phi_\beta^*, \quad (4.14)$$

first proposed by Weinberg in [38]. Note that to obtain light neutrino masses of  $m_\nu \ll 1$  eV, the right-handed mass scale will need to be  $M_N \gtrsim 10^{14}$  GeV, which is surprisingly close to the scales of unification seen in MSSM and  $SU(5)$ .

The alternative case for neutrino mass that includes the left-handed Majorana term, as seen above in the left-right model case, and as will be the case for  $SO(10)$ , is known as the *type-II seesaw*. The corresponding light neutrino masses for this case will generally be of the form

$$m_\nu \simeq f v_L - \frac{y_\nu^2 v^2}{f v_R}, \quad (4.15)$$

with  $v_L \sim v^2/v_R$ . Note that generally the type-I term will be present in the type-II case, although one may see dominance of either term depending on the couplings and the scale of  $v_R$ . One can implement type-II seesaw through extension of the SM as well, by for instance adding a heavy triplet  $\Delta_L$  with couplings of the form  $\ell^T \sigma_2 \Delta_L \ell$  and  $\phi^T \sigma_2 \Delta_L \phi$ , which gives rise to an effective operator similar to that in (4.14). Other forms are plausible as well but typically require more highly *ad hoc* or tuned assumptions.

## 4.3 $SO(10)$ Grand Unification

### 4.3.1 Representations of $SO(N)$ and $SO(2N)$

For the N-dimensional fundamental representation of the group  $SO(N)$ , one can define a basis in the conventional way,

$$(J^{ab})_{mn} \equiv -i \delta_{[m}^a \delta_{n]}^b = -i (\delta_m^a \delta_n^b - \delta_n^a \delta_m^b)$$

such that the Lie algebra bracket condition

$$[J_{ab}, J_{cd}] = -i (\delta_{b[c} J_{ad]} + \delta_{a[d} J_{bc]}) \quad (4.16)$$

is satisfied. These generators are of course analogous to the usual angular momentum generators in  $SO(3)$ ; thus, I can write the orthogonal transformation (*i.e.*, length-preserving rotation) of an N-dimensional vector  $V_m$  as

$$V_m \rightarrow O_{mn} V_n = \exp \left\{ -\frac{i}{2} \theta_{ab} (J^{ab})_{mn} \right\} V_n.$$

Tensor representations of larger dimensions can be constructed in the usual way

$$T_{mn\dots} = V_m \otimes W_n \otimes \dots$$

In addition to fundamental and tensor representations,  $SO(2N)$  will have a spinor representation<sup>2</sup> in its universal covering group  $Spin(2N)$ , and the Lie algebras of the two groups will be isomorphic. In Euclidean analogy to the Dirac algebra of the Lorentz group, the objects  $\Gamma_m$ , with  $m = 1, \dots, 2N$ , are  $2^N \times 2^N$  matrices that satisfy the Clifford algebra condition

$$\{\Gamma_m, \Gamma_n\} = 2\delta_{mn}\mathbb{I}_{2^N} \quad (4.17)$$

and act on  $2^N$ -dimensional spinors  $\psi$ . If I define

$$\Sigma_{mn} \equiv -\frac{i}{4} [\Gamma_m, \Gamma_n], \quad (4.18)$$

one finds that the  $\Sigma_{mn}$  satisfy the  $SO(2N)$  algebra (4.16) and are therefore a valid representation of the group. I can write the transformation of a spinor  $\psi_\alpha$  as

$$\psi_\alpha \rightarrow U_{\alpha\beta} V_\beta = \exp \left\{ -\frac{i}{2} \theta_{mn} (\Sigma_{mn})_{\alpha\beta} \right\} V_\beta.$$

In analogy with  $\gamma_5$  of the Dirac algebra, the object

$$\Gamma_0 \equiv i^{2N} \Gamma_1 \Gamma_2 \dots \Gamma_{2N}$$

allows for projection of the  $2^N$ -dimensional spinor into two  $2^{N-1}$ -dimensional chiral components by

$$\psi_{L,R} = \frac{1}{2} (1 \pm \Gamma_0) \psi. \quad (4.19)$$

Also of interest is the  $Spin(2N)$  basis as an extension of an  $SU(N)$  basis. If one takes the complex operators  $\chi_a$ , for  $a = 1, 2, \dots, N$  satisfying

$$\{\chi_a, \chi_b^\dagger\} = \delta_{ab},$$

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<sup>2</sup>One can of course construct a spinor representation for  $SO(N)$  with  $N$  odd as well, though it requires a bit more consideration.

then the operators  $T_{ab} = \chi_a^\dagger \chi_b$  satisfy the  $\mathfrak{su}(N)$  Lie algebra, while the operators

$$\begin{aligned}\Gamma_{2a} &\equiv (\chi_a + \chi_a^\dagger) \\ \Gamma_{2a-1} &\equiv -i(\chi_a - \chi_a^\dagger)\end{aligned}\tag{4.20}$$

are 2N objects satisfying the Clifford algebra in (4.17), and therefore form a valid representation for  $\Gamma_m$ .

### 4.3.2 The Basics of $SO(10)$ as an Interacting Gauge Theory

Following the prescription above, the rank-5 simple group  $SO(10)$  has a **16**-dimensional Weyl-spinor representation in its covering group  $Spin(10)$ ; <sup>3</sup> the **16** decomposes in  $SU(5) \times U(1)$  as  $\mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1}$ ; given the matter field content of the  $SU(5)$  representations, this decomposition is highly suggestive. Taking the  $SU(5)$  representations as usual and the right-handed neutrino as the singlet, one sees that all matter fermions and anti-fermions of a single generation and chirality fit exactly into one chiral  $SO(10)$  spinor, denoted by  $\psi_{L,R}$ . Since the anti-particle fields of some chirality correspond to the particle fields of opposite chirality, one finds all of the left- and right-handed fields in a single chiral spinor. Therefore, in building an  $SO(10)$  model, I have no need for the full 32-dimensional spinor, and I will simply denote the chiral spinor by  $\psi$ , which I assume left-handed by convention.

The explicit arrangement of the field content in  $\psi$  depends on the choice of basis for the generators  $\Sigma_{mn}$ , and hence the choice of basis for  $\Gamma_m$  ( $m = 1, 2, \dots, 10$ ), for which there are many. The end result is quite tedious not of much use other than for explicit calculation. The kinetic term for  $\psi$ , however, can nonetheless be written in a familiar form:

$$\mathcal{L}_{U,\text{kin}} = \bar{\psi} i \not{D} \psi = \bar{\psi} \gamma_\mu \left( i \partial_\mu + \frac{g_U}{2} \Sigma_{mn} W_\mu^{mn} \right) \psi; \tag{4.21}$$

the matrix  $(\Sigma_{mn} W_\mu^{mn})_{ab}$  is generally  $32 \times 32$  in spin space but will be block diagonal and redundant for reps based on the **16** spinor.  $W_\mu^{mn}$  are the 45 gauge bosons of the

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<sup>3</sup>In keeping with convention, I will often refer to this representation as the “ $SO(10)$  spinor” rep.

model (*i.e.*,  $\binom{10}{2}$ ), which decompose under the SM gauge group as

$$\begin{aligned}
\{45\} = & G(\mathbf{8}, \mathbf{1}, 0) \oplus W_L(\mathbf{1}, \mathbf{3}, 0) \oplus X_{B-L}(\mathbf{1}, \mathbf{1}, 0) \\
& \oplus X_\rho^{u,d}(\mathbf{3}, \mathbf{2}, -\frac{5}{6}) \oplus \bar{X}_{\bar{\rho}}^{u,d}(\bar{\mathbf{3}}, \mathbf{2}, \frac{5}{6}) \oplus Y_\rho^{u,d}(\mathbf{3}, \mathbf{2}, \frac{1}{6}) \oplus \bar{Y}_{\bar{\rho}}^{u,d}(\bar{\mathbf{3}}, \mathbf{2}, -\frac{1}{6}) \\
& \oplus A_\rho(\mathbf{3}, \mathbf{1}, \frac{1}{3}) \oplus \bar{A}_{\bar{\rho}}(\bar{\mathbf{3}}, \mathbf{1}, -\frac{1}{3}) \\
& \oplus W_R^+(\mathbf{1}, \mathbf{1}, \frac{1}{2}) \oplus W_R^-(\mathbf{1}, \mathbf{1}, -\frac{1}{2}) \oplus W_R^3(\mathbf{1}, \mathbf{1}, 0);
\end{aligned}$$

when compared to  $SU(5)$ , one might notice that (a) the diagonal hypercharge generator has been swapped for the  $B - L$  generator and that of the neutral right-handed  $W_R^3$ , thereby increasing the rank of the group by one, as expected, and (b) another set of bosons  $Y$  with both color and  $\mathbf{T}_L$  weak isospin are present, in addition to the  $X$  bosons of  $SU(5)$ . In fact, both the  $X$  and  $Y$  bosons have  $\mathbf{T}_R$  isospin as well here, and pair off cross-wise under  $SU(2)_R$ , as  $(Y^u, X^u)_\rho$ ,  $(\bar{Y}^d, \bar{X}^d)_{\bar{\rho}}$ , etc. For a complete analysis of the bosons, their corresponding generators, and their decompositions in several bases and for several subgroups, see, *e.g.*, [39].

### 4.3.3 Fermion Masses and Higgs Representations in $SO(10)$

Because particles and anti-particles in  $SO(10)$  are together in the same chiral spinor, generating mass terms requires additional complexity when compared to the familiar low-energy theory. In particular, one sees non-trivial algebraic structure in the Yukawa couplings.

The tensor product of two chiral spinors decomposes in the group as  $\mathbf{16} \otimes \mathbf{16} = \mathbf{10} \oplus \mathbf{120} \oplus \bar{\mathbf{126}}$ ; the  $\mathbf{10}$  and  $\mathbf{120}$  are the fundamental rep and the 3-index totally anti-symmetric rep, respectively, and the 5-index, totally anti-symmetric rep  $\mathbf{252}$  decomposes into  $\mathbf{126} \oplus \bar{\mathbf{126}}$ . Therefore one expects the Yukawa couplings of Higgs fields to matter in the model to appear in one of the three above representations.

In the simplest case, an  $SO(10)$  model has only a  $\mathbf{10}$ -dimensional Higgs field  $H_m$ ; its coupling to  $\psi\psi$  has the explicit form

$$\mathcal{L}_{U,Yuk} \ni h_{ij} \psi_i^T B C^{-1} \Gamma_m \psi_j H_m, \quad (4.22)$$

where the Yukawa coupling  $h_{ij}$  is symmetric in the generation space. The matrix  $B$  appearing here plays a role analogous to that of  $C$  but in the  $Spin(10)$  space: under

the spin group, the spinor  $\psi$  and its conjugate transform as

$$\delta\psi = i\omega_{mn}\Sigma_{mn}\psi \quad \delta\psi^\dagger = -i\omega_{mn}\psi^\dagger\Sigma_{mn},$$

where I've used that the generators  $\Sigma_{mn}$  are Hermitian; however,

$$\delta\psi^T = i\omega_{mn}\psi^T\Sigma_{mn}$$

does not transform like a conjugate field. Therefore, one defines the matrix  $B$  such that

$$\delta(\psi^T B) = -i\omega_{mn}(\psi^T B)\Sigma_{mn}.$$

Explicitly,  $B$  can be given as  $B \equiv \Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_9$ , which further implies that

$$B^{-1}\Gamma_m B = -\Gamma_m.$$

As in  $SU(5)$  and the SM, I want a vev for  $H$  to break  $SU(2)_L$  in order to give the fermions mass. Looking at eq. (4.20), note that for the fields  $\chi^a$ , the components  $a = 1, 2, 3$  relate to color, while  $a = 4, 5$  relate to left isospin. I will take the vev to correspond to  $a = 5$ , which implies  $\langle H_9, H_{10} \rangle \neq 0$ . If I take  $\langle H_9 \rangle = v_1$  and  $\langle H_{10} \rangle = v_2$ , then one finds the following terms for fermion masses (considering a single generation for now):

$$\mathcal{L}_{\text{Yuk}, \mathbb{H}} = h(v_2 - v_1) (\bar{d}_L d_R + \bar{e}_L e_R) + h(v_2 + v_1) (\bar{u}_L u_R + \bar{\nu}_L \nu_R) + \text{h.c.s};$$

this result implies  $m_e = m_d$  and  $m_u = m_\nu$ . Although this is a GUT-scale result, it cannot be made to agree with low-energy observations, even when running effects are taken into account. This is even more strongly the case for second generation; hence, to build a realistic model, one needs additional Higgs Yukawas.

The next available option for Higgs field is the **120**-dimensional field  $\Sigma_{mno}$ , which couples to the fermions by

$$\mathcal{L}_{\text{U, Yuk}} \ni g_{ij} \psi_i^T B C^{-1} \Gamma_m \Gamma_n \Gamma_o \psi_j \Sigma_{mno}; \quad (4.23)$$

the Yukawa coupling matrix  $g_{ij}$  is anti-symmetric in order to preserve  $SO(10)$  invariance; therefore, this Yukawa can only contribute to mass mixing among generations.

There are several potential vevs that do not disturb color invariance. If I choose



$\langle \Sigma_{789}, \Sigma_{780} \rangle \neq 0$  (I will use “0” instead of “10” for multi-index fields to avoid confusion), then the resulting mass relationships are

$$m_d^i = 3m_e^{ij}, \quad m_u^i = 3m_\nu^{ij};$$

*i.e.*, the contribution to the  $(ij)$ -element of electron mass matrix is proportional to the  $i^{th}$  down mass, and similar for the up-type particles. Clearly this Higgs field would need to be used in conjunction with others to achieve a realistic mass spectrum.

The final choice for a Higgs is the  $\overline{126}$  field  $\bar{\Delta}_{mnopq}$ ; its coupling to the fermions is

$$\mathcal{L}_{U,Yuk} \ni f_{ij} \psi_i^T B C^{-1} \Gamma_m \Gamma_n \Gamma_o \Gamma_p \Gamma_q \psi_j \bar{\Delta}_{mnopq}, \quad (4.24)$$

where  $f_{ij}$  is symmetric. The following vevs preserve  $SU(3)_C$ :

$$\langle \bar{\Delta}_{1278m} \rangle = \langle \bar{\Delta}_{3478m} \rangle = \langle \bar{\Delta}_{5678m} \rangle \neq 0, \quad m = 9 \text{ or } 10,$$

which give the mass relations

$$m_e^{ij} = -3m_d^{ij}, \quad m_\nu^{ij} = -3m_u^{ij};$$

this result nicely predicts the observed  $\frac{m_e}{m_\mu} : \frac{m_d}{m_s}$  ratio, but does not agree with third generation observations. A realistic mass spectrum can though be obtained through a combination of H and  $\bar{\Delta}$ .

The  $\overline{126}$  Higgs may play another important role in the fermion mass spectrum. Under decomposition to left-right models, the field contains a right-handed triplet part. A vev for this component breaks  $B - L$ , and it couples to  $\nu_R \nu_R$  as in eq. (4.2); furthermore, the field corresponds to the  $SU(5)$  singlet, so it does not disturb  $SU(3)_C \times SU(2)_L$ . Hence, if this triplet acquires a vev around the GUT scale, it will simultaneously explain the suppression of right-handed currents and activate the type-I seesaw for neutrino mass.

#### 4.3.4 Spontaneous Symmetry Breaking in $SO(10)$

$SO(10)$  has two maximal subgroups of relevance to symmetry breaking:

$$SO(10) \supseteq SU(5) \times U(1), \quad SO(6) \times SO(4);$$

The context of the former should be clear, as it has been mentioned previously. To understand the significance of the latter decomposition, note that

$$Spin(6) \cong SU(4), \quad Spin(4) \cong SU(2)_- \times SU(2)_+;$$

hence,  $Spin(6) \times Spin(4) \cong$  Pati-Salam (PS); more specifically, the breakdown of  $SO(10)$  to PS is

$$Spin(10) \longrightarrow SU(2)_L \times SU(2)_R \times SU(4)_C \times \mathbb{Z}_2;$$

in full  $SO(10)$  representations, the  $\mathbb{Z}_2$  symmetry is manifested as *D-parity* [40]; the explicit form of a *D-parity* transformation is

$$\begin{aligned} D(V_m) &\equiv \exp(-i\pi J_{23}) \exp(i\pi J_{67}) \\ D(\psi) &\equiv \exp(-i\pi \Sigma_{23}) \exp(i\pi \Sigma_{67}) = -\Gamma_2 \Gamma_3 \Gamma_6 \Gamma_7, \end{aligned}$$

which corresponds to a pair of  $\pi$ -rotations in the (23) and (67) planes of the 10-dimensional vector space of the fundamental. Since the matter field  $\psi$  contains only fields of a single chirality, there can be no well-defined notion of parity in  $SO(10)$ ; *D-parity* then plays a role to create to the possibility for the presence of *C* and *P* at lower energies.

As I mentioned earlier, the matter spinor decomposes under  $SU(5) \times U(1)$  as  $\mathbf{16} = \mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1}$ ; under Pati-Salam, the decomposition makes “left-right” splitting manifest:  $\mathbf{16} = (\mathbf{2}, \mathbf{1}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}})$ , but let me reiterate that right-handed fields are still explicitly absent; for example, the doublet one might be inclined to call “ $q_R$ ” is actually  $q_L^c$ . In breaking  $SO(10)$  to Pati-Salam, the  $\mathbb{Z}_2$  coming from conservation of *D-parity* corresponds to  $\mathbf{2}_L \leftrightarrow \mathbf{2}_R$  under charge conjugation symmetry. Hence one finds Pati-Salam with “left-right” symmetry, in the sense that  $g_{2L} = g_{2R}$ , but nonetheless defined with left-handed antiparticle fields rather than right-handed particle fields.

For either class of breaking possibilities, one must of course consider only vevs which leave  $SU(3)_C \times U(1)_{\text{em}}$  unbroken; furthermore, since one expects to find that group as a consequence of breaking the usual SM gauge group, further restriction to vevs which leave  $SU(2)_L$  in tact is also needed. Note that in general the Higgs fields with components that acquire vevs will not be those that couple to matter; *i.e.*, additional representations of Higgs may be present in the scalar potential of the  $SO(10)$  model,

coupled only to other Higgs fields.

**$SO(10) \rightarrow SU(5)$ .** To induce the breaking of  $SO(10)$  to  $SU(5)$ , one simply gives a vev to the  $SU(5)$ -singlet component of some appropriate Higgs, which usually also breaks  $B - L$ . Two such choices are the **1** of a  **$16_H$**  or  **$126$** . The 2-index, totally anti-symmetric **45** rep of  $SO(10)$  contains the **24** of  $SU(5)$ , so if one includes that field, the breaking of  $SU(5) \rightarrow \text{SM}$  proceeds as discussed in section 4.1.2.

Assuming  $SO(10)$  breaks at the GUT scale,  $M_U \sim 2 \times 10^{16} \text{ GeV}$  and  $SU(5)$  breaks at its canonical scale of  $M_X \sim 10^{14-15}$ , this model would be ruled out by proton decay constraints; hence any applications of these breaking patterns would need to be at higher scales in more elaborate models.

**$SO(10) \rightarrow \text{PS \& Left-Right}$ .** Breaking  $SO(10)$  to the Pati-Salam gauge group is a considerably more fruitful choice, with not only many choices for path of breaking, but also the possibility for robust intermediate scale physics, because left-right symmetric models are phenomenologically eligible for breaking at scales as low as 1 TeV, although doing so sacrifices the possibility for implementing the seesaw mechanism specifically as described in section 4.2.

Some of the most common vev choices for breaking to PS include the **(1, 1, 1)** component of the 2-index, traceless symmetric **54** rep and the **(1, 1, 1)** or **(1, 1, 15)** component of the 4-index, anti-symmetric **210** rep. The **54** option preserves  $D$ -parity, while the **210** choices do not. In the **54** case, one can further break to  $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$  through the **(1, 1, 15)** component of **45**, which also breaks  $D$ -parity.

In all of the cases described above, breaking to the SM requires  $SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y$ ; in the PS cases, one must also break  $SU(4)_C$  as well, but since  $SU(4)_C \supseteq SU(3)_C \times U(1)_{B-L}$ , the breaking of  $B - L$  will accomplish both tasks.<sup>4</sup> The most common approaches involve vevs for either the **(1, 3, 10)** component of  **$\overline{126}$** , denoted  $\bar{\Delta}_R$ , or the singlet of  **$16_H$** . The  **$\overline{126}$**  case has clear advantages over that of  **$16_H$** :

- One can see from the PS representation of  $\bar{\Delta}_R$  that it is a right-handed triplet, which is precisely the object present in the right-handed Majorana neutrino mass term in eq. (4.2). Hence the vev  $\langle \bar{\Delta}_R \rangle \equiv v_{B-L} = v_R$ , and implementation of

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<sup>4</sup>One can instead break only  $SU(2)_R \rightarrow U(1)_R$  if looking to leave  $SU(4)_C$  (and hence  $B - L$ ) in tact.

the seesaw mechanism comes for free from the  $B - L$  breaking; this attractive scenario of a single mechanism performing two crucial duties in the model is quite economical to say the least. Furthermore, the  $\overline{\mathbf{126}}$  coupling  $f_{ij}$  will be highly constrained by the mass spectrum of the charged fermions, and yet will be present in the Majorana neutrino terms also; so the economy of the model extends to its number of parameters as well.

In contrast, one must include higher dimensional operators or singlet fields to obtain the  $\nu^c$  mass term in the case with  $\mathbf{16}_H$ .

- The  $\bar{\Delta}_R$  breaks of  $B - L$  by two units in the emergence of the  $\nu^c \nu^c$  mass term. Note that for a supersymmetric model, this leaves  $R$ -parity,  $R = (-1)^{3(B-L)+2s}$ , conserved. This is of course attractive if one would like to suppress  $R$ -parity violating terms and retain the potential for an LSP dark matter candidate.

The  $\mathbf{16}_H$  field, however, corresponds to the  $\nu^c$  component and therefore breaks  $B - L$  by a single unit, which is  $R$ -parity odd. As a result, one finds  $R$ -parity violating terms among the higher dimensional operators involving  $\mathbf{16}_H$ .

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The procedure for constructing a properly broken subgroup at some scale requires several steps when considering larger groups such as  $SO(10)$ , especially in the rank-reducing cases. First, one must rescale all the generators for the “before” and “after” groups such that they share a common normalization. Next, for a breaking of the form  $G_1 \times G_2 \longrightarrow G_0$  at energy scale  $M$ , where generators  $T_1$  and  $T_2$  will merge in the breaking as

$$T_0 = a_1 T_1 + a_2 T_2,$$

then the corresponding gauge couplings  $g_1, g_2, g_0$  must satisfy the following boundary condition:

$$\frac{1}{\alpha_0(M)} = \frac{a_1^2}{\alpha_1(M)} + \frac{a_2^2}{\alpha_2(M)}, \quad (4.25)$$

where  $\alpha_i = g_i^2/4\pi$  is the fine structure constant for the group  $G_i$ . Finally, one must consider the running of each coupling between the various scales. In particular, the evolution of  $\alpha_i$  between two mass scales  $M_2 > M_1$  follows from the RGE for the coupling:

$$\frac{1}{\alpha_i(M_1)} = \frac{1}{\alpha_i(M_2)} - \frac{b_i}{2\pi} \ln \left( \frac{M_2}{M_1} \right), \quad (4.26)$$

where  $b_i$  are model and group-specific beta function coefficients discussed in section 3.2.2. Note that in cases involving multi-step breaking patterns and multiple couplings, these relationships will be used iteratively. In this manner, one can develop the precise relationships between low-scale measured parameters and (heavy:light) mass scale ratios, which can be used to experimentally test GUT models, set lower limits on heavy scales, etc. One pertinent example is the ability to constrain GUTs using the experimental limits on  $\sin^2 \theta_W = \alpha_{\text{em}}/\alpha_{2L}$  combined with the higher order corrections to its value coming from the relationship in (4.26).

### 4.3.5 Supersymmetry and $SO(10)$

Since some of the unresolved issues of the SM are obviated by SUSY, and some others are successfully attended to by  $SO(10)$  unification, it would seem quite wise to consider the merging of the two frameworks into a SUSY  $SO(10)$  model of the universe. Most clearly of importance is that non-SUSY GUT models face the problems with quadratic divergences in loop corrections to Higgs masses. In addition to the benefits coming from one framework or the other, a few added benefits arise from the combination, including possible restrictions of soft  $CP$  phases in SUSY, similar constraint of the strong  $CP$  phase, and, as I mentioned in the previous section, the possibility of automatic  $R$ -parity conservation.

The promotion of  $SO(10)$  to a supersymmetric model follows quite straightforwardly from the process for constructing the MSSM; in particular, the SM fermion content is unchanged (other than the addition of the right-handed neutrino, of course), and all of the same formalism applies for new scalar and gauge boson superpartners, auxiliary fields, etc.

One caveat does arise with respect to vevs for the various Higgs fields: for any field with a vev that reduces the rank of the group, one must include the barred partner for the field, so that their D-terms in the scalar potential cancel with each other; this keeps SUSY unbroken above the desired scale, which is thought to be  $\mathcal{O}(\text{TeV})$ . In particular, the breaking  $SU(2)_R \times U(1)_{B-L} \rightarrow U(1)_Y$  will require  $\overline{\mathbf{126}} + \mathbf{126}$  or  $\mathbf{16}_H + \overline{\mathbf{16}}_H$ .

As an example, consider the well-known “minimal” SUSY  $SO(10)$  model, which includes  $\mathbf{10}$  and  $\overline{\mathbf{126}}$  Higgs fields coupling to matter plus a  $\mathbf{210}$  field to initiate the GUT scale breaking. Yukawa terms in the superpotential for can be written by simply promoting the fermionic matter spinors and Higgs scalars in eqs. (4.22) and (4.24) to superfields; the remaining terms will be all quadratic or cubic superfield products al-

lowed by the  $SO(10)$  invariance, of the form in eq. (3.35). The resulting superpotential for the this model, up to  $\mathcal{O}(1)$  numerical factors, is

$$\begin{aligned}
W_U = & M_{210} \hat{\Phi}^2 + \lambda \hat{\Phi}_{lmno} \hat{\Phi}_{nopq} \hat{\Phi}_{pqlm} + M_{10} \hat{\mathbf{H}}^2 + M_{126} \hat{\Delta} \hat{\bar{\Delta}} \\
& + \eta \hat{\Phi}_{lmno} \hat{\Delta}_{lmnpqr} \hat{\bar{\Delta}}_{nopqr} + \hat{\mathbf{H}}_l \hat{\Phi}_{mnop} \left( \gamma \hat{\Delta}_{lmnop} + \bar{\gamma} \hat{\bar{\Delta}}_{lmnop} \right) \\
& + h_{ij} \Psi_i B \Gamma \Psi_j \hat{\mathbf{H}} + f_{ij} \Psi_i B \Gamma \Gamma \Gamma \Gamma \Psi_j \hat{\bar{\Delta}},
\end{aligned} \tag{4.27}$$

where  $i, j = 1, 2, 3$  are the generation indices,  $l, m, n, \dots = 1, \dots, 10$  are  $SO(10)$  indices, and I have suppressed the  $SO(10)$  indices for straightforward contractions. Here I have used hats in the denotations of the Higgs superfields to distinguish them from their scalar components; otherwise, my notation conventions from Chapter 3 for denoting superfields and their components will remain in tact for the rest of this work.

One more point of interest is that any Higgs superfield in the theory in an  $SU(2)_L \times SU(2)_R$  bi-doublet representation, *i.e.*, with PS quantum numbers  $(\mathbf{2}, \mathbf{2}, x)$ , that also breaks to an  $SU(3)_C$  singlet will contribute to the linear combinations which remain light and play the roles of  $H_{u,d}$  at the electroweak scale. Contributions will generally come even from components which do not couple to matter, through mixing with those that do, once vevs are acquired. I will discuss this topic in more detail in the next section, where I will give the details of the model on which this work is based.

## 4.4 A SUSY $SO(10)$ Model of Unification

The SUSY  $SO(10)$  model on which my proton decay analysis is based has **10**,  $\overline{\mathbf{126}}$ , and **120** Higgs superfields with Yukawa couplings contributing to fermion masses; denotation of each is consistent with the previous section. The superpotential for the model is given by eq. (4.27) plus the following additional terms due to the presence of the **120** field:

$$\begin{aligned}
W_U \ni & M_{120} \hat{\Sigma}^2 + \kappa \hat{\Sigma}_{mno} \hat{\mathbf{H}}_p \hat{\Phi}_{pmno} + \rho \hat{\Sigma}_{lmp} \hat{\Sigma}_{nop} \hat{\Phi}_{lmno} \\
& + \hat{\Sigma}_{lmn} \hat{\Phi}_{nopq} \left( \zeta \hat{\Delta}_{opqlm} + \bar{\zeta} \hat{\bar{\Delta}}_{opqlm} \right) + g_{ij} \Psi_i B \Gamma \Gamma \Gamma \Psi_j \hat{\Sigma},
\end{aligned} \tag{4.28}$$

where again  $i, j = 1, 2, 3$  are the generation indices, and I have suppressed the  $SO(10)$  indices for total contractions. Here  $\Psi_i$  is the **16**-dimensional matter spinor containing chiral superfields for all the SM fermions (of one generation) plus the left-handed anti-

neutrino.

**Type-I Seesaw Breaking Pattern.** For the type-I seesaw implementation, breaking of  $SO(10)$  to MSSM proceeds as follows:

$$\begin{aligned}\langle \Phi(\mathbf{1}, \mathbf{1}, \mathbf{1}) \rangle : SO(10) &\longrightarrow SU(4)_C \times SU(2)_L \times SU(2)_R, \quad (\not{D}) \\ \langle \bar{\Delta}(\mathbf{1}, \mathbf{3}, \mathbf{10}) \rangle \equiv v_R : SU(4)_C \times SU(2)_L \times SU(2)_R &\longrightarrow \text{MSSM}.\end{aligned}$$

Note that  $\langle \Delta(\mathbf{1}, \mathbf{3}, \overline{\mathbf{10}}) \rangle = v_R$  is also present such that D-term contributions will cancel. The value of  $\langle \Phi \rangle$  is taken at the coupling unification scale  $M_U \sim 2 \times 10^{16}$  GeV, and  $v_R$  at  $\sim 10^{15}$  GeV; hence any running under PS is negligible. As discussed previously, the  $\hat{\Delta}_R$  component superfield couples to the right-handed neutrino  $\mathcal{N}^c$ . Thus the acquisition of the vev  $v_R$  will lead to the Majorana mass term

$$W_{\mathcal{N}} \ni f_{ij} \hat{\Delta}_R \mathcal{N}^c \mathcal{N}^c \xrightarrow{\langle \bar{\Delta}_R \rangle} f v_R \nu_R^T C^{-1} \nu_R; \quad (4.29)$$

furthermore, this term will induce a type-I seesaw mass for  $\nu_L$  after EWSB:

$$m_\nu = -\frac{y_\nu^2 v^2}{f v_R}. \quad (4.30)$$

**Type-II Seesaw Breaking Pattern.** The coupling to matter of the left-handed PS (and SM) triplet  $\hat{\Delta}_L \equiv \hat{\Delta}(\mathbf{3}, \mathbf{1}, \overline{\mathbf{10}})$  as seen in eq. (4.2) is present in any model with a  $\overline{\mathbf{126}}$  field; hence, to give a type-II Majorana mass to the neutrino, one simply must give a vev to the scalar  $\langle \bar{\Delta}_L \rangle \equiv v_L$ . That said, the only motivation for giving such an extremely tiny vev,  $\mathcal{O}(10^{-2} \text{ eV})$ , is strictly empirical. However, if the vev for  $v_L$  were instead inversely related to a heavy scale already present in the theory, then its small value would be nicely consistent. In order to create such a scenario, the most straightforward option is to include a **54** multiplet  $\hat{\mathbf{S}}_{mn}$  in the Higgs spectrum. This field adds the following pertinent terms to the superpotential (among others not important here):

$$W_U \ni \xi \hat{\mathbf{S}}_{mn} \hat{\mathbf{H}}_m \hat{\mathbf{H}}_n + \eta' \hat{\mathbf{S}}_{lm} \hat{\Delta}_{lnopq} \hat{\Delta}_{mnopq} + \bar{\eta}' \hat{\mathbf{S}}_{lm} \hat{\Delta}_{lnopq} \hat{\Delta}_{mnopq}. \quad (4.31)$$

The F-term of  $\hat{\mathbf{S}}$  then gives rise to a scalar operator of the form

$$\left[ W(\hat{\mathbf{S}}) \right]_F \ni \xi \bar{\eta}' H_u \bar{\Delta}_L H_u \bar{\Delta}_R,$$

which will consequently appear in the F-term for  $\hat{\bar{\Delta}}_L$  as well, leading to the scalar potential

$$V(\bar{\Delta}_L) = M_{126}^2 |\bar{\Delta}_L|^2 + \xi \bar{\eta}' H_u \bar{\Delta}_L H_u \bar{\Delta}_R;$$

now the vevs for  $h_u^0$  and  $\bar{\Delta}_R$  will induce a vev for  $\hat{\bar{\Delta}}_L$  of the form

$$v_L \equiv \langle \bar{\Delta}_L \rangle = \frac{\xi \bar{\eta}' v_u^2 v_R}{M_{126}^2} \sim \frac{1}{M_U}.$$

Thus the full low-scale neutrino mass matrix becomes

$$m_\nu = f v_L - \frac{y_\nu v^2 y_\nu^T}{f v_R}. \quad (4.32)$$

However, an examination of this expression in light of the values for the various parameters will reveal that the type-I and type-II contributions in (4.32) are generally comparable. Hence this prescription is not enough on its own to give type-II dominance. To induce a truly dominant type-II seesaw, one needs additional structure to somehow decouple the mass of  $\bar{\Delta}_L$  from that of  $\bar{\Delta}$ .

One particularly nice way to accomplish this, which was first discussed in [41], goes as follows. One first breaks  $SO(10)$  together with  $B - L$  by giving a vev to  $\bar{\Delta}_R$  at a scale  $\gtrsim 10^{17}$  GeV, resulting in  $SU(5)$ ; here, the left-handed triplet  $\hat{\bar{\Delta}}_L$  is part of the two-index symmetric **15** representation. Generally the **15** components coming from the  $\overline{\mathbf{126}}$ , **126**, and **210**, will have comparable masses. The vev for  $\hat{\bar{\Delta}}_L \sim 1/M_{126}$ , so for larger  $v_L \sim \mathcal{O}(\text{eV})$ , one would like to lower the scale to  $M_{126} \lesssim 10^{13}$  GeV; however, the decomposition of  $\overline{\mathbf{126}}$  gives rise to additional  $SU(5)$  reps such as **45** and  $\overline{\mathbf{50}}$ , which also have masses  $\sim M_{126}$ ; if all such multiplets become so light, gauge coupling unification will be irreparably damaged. The day is saved, though, by the presence of the **54** Higgs  $\hat{\mathbf{S}}$ , which decomposes under  $SU(5)$  as  $\mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{24}$ , and thus contributes to the **15** mass matrix but not those of **45** and **50**. As a result, the masses for **15** can be tuned to the required light scale without other consequences, and the vev for  $\bar{\Delta}_L$

$$v_L = \frac{f \xi \bar{\eta}' v_u^2}{M_{\bar{\Delta}_L}}$$

can be larger as needed for type-II dominance.

With the light mass for  $\bar{\Delta}_L$  on hand, one breaks  $SU(5)$  at the usual coupling unification scale by  $\Phi(\mathbf{24}) \in \mathbf{210}$ ; hence, the  $SO(10)$  breaking chain for type-II dominance



is

$$\begin{aligned}\langle \bar{\Delta}(\mathbf{1}) \rangle : SO(10) &\longrightarrow SU(5), \ (\not{D}) \\ \langle \Phi(\mathbf{24}) \rangle : SU(5) &\longrightarrow \text{MSSM};\end{aligned}$$

I've used notation for the  $SU(5)$  reps here, but note that the two components present correspond precisely to those acquiring vevs in the type-I case.

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After breaking to MSSM,  $SU(2)_L$  doublets with SM quantum numbers  $((\mathbf{1}, \mathbf{2}, -\frac{1}{2}) + \text{c.c.})$ , which have their origins in the PS bi-doublet components  $\hat{\mathbf{H}}(\mathbf{2}, \mathbf{2}, \mathbf{1})$ ,  $\hat{\Delta}(\mathbf{2}, \mathbf{2}, \mathbf{15})$ , and  $\hat{\Sigma}(\mathbf{2}, \mathbf{2}, \mathbf{1}) + \hat{\Sigma}(\mathbf{2}, \mathbf{2}, \mathbf{15})$ , have the following couplings to matter superfields in the superpotential:

$$\begin{aligned}W_{\text{Yuk}} = & h\epsilon_{ab} \left\{ \hat{\mathbf{H}}_u^b (Q^a U^c + L^a \mathcal{N}^c) + \hat{\mathbf{H}}_d^b (Q^a D^c + L^a E^c) \right\} \\ & + \frac{f\epsilon_{ab}}{\sqrt{3}} \left\{ \hat{\Delta}_u^b (Q^a U^c - 3L^a \mathcal{N}^c) + \hat{\Delta}_d^b (Q^a D^c - 3L^a E^c) \right\} \\ & + g\epsilon_{ab} \left\{ \hat{\Sigma}_u^{1b} (Q^a U^c + L^a \mathcal{N}^c) + \hat{\Sigma}_d^{1b} (Q^a D^c + L^a E^c) \right\} \\ & + \frac{g\epsilon_{ab}}{\sqrt{3}} \left\{ \hat{\Sigma}_u^{15b} (Q^a U^c - 3L^a \mathcal{N}^c) + \hat{\Sigma}_d^{15b} (Q^a D^c - 3L^a E^c) \right\},\end{aligned}\quad (4.33)$$

where I've suppressed generation and color indices. As one can see, these doublets come in pairs with opposite hypercharge and so have the form of the SUSY Higgs doublets  $H_{u,d}$ . Furthermore, these fields will mix with one another, and also with doublets from **126** and **210**, to form mass eigenstates. If I take all such component fields in the obvious basis as

$$\varphi_u \equiv \left( \hat{\mathbf{H}}_u, \hat{\Sigma}_u^1, \hat{\Sigma}_u^{15}, \hat{\Delta}_u, \hat{\Delta}_u, \hat{\Phi}_u \right),$$

and similar for  $\varphi_d$ , but with  $\hat{\Delta}_u \rightarrow \hat{\Delta}_d$  and “vice versa”, then the mass matrix  $\mathcal{M}_{\mathcal{D}}$  is defined such that the mass states are given by  $\varphi_d^T \mathcal{M}_{\mathcal{D}} \varphi_u$ ; the form of  $\mathcal{M}_{\mathcal{D}}$  can be seen in [42]. The matrix is diagonalized by a bi-unitary transformation  $\mathcal{U} \mathcal{M}_{\mathcal{D}} \mathcal{V}^T$ , giving the mass eigenstates for the doublet superfields as linear combinations of the component fields. Note that this matrix is fully determined by the couplings and vevs of the superpotential (although the majority of those parameters are virtually unconstrained), and so the fields are generally expected to be heavy; however, one

doublet pair must remain light in order to play the role of the MSSM Higgs doublets  $H_{u,d}$ . This point requires the imposing of the condition  $\text{Det } \mathcal{M}_{\mathcal{D}} \sim 0$  (*i.e.*,  $M_{\text{SUSY}} \sim 0$  when compared to the GUT scale), which can be realized by fine-tuning one of the parameters in the matrix, conventionally chosen to be the mass of  $\hat{\mathbf{H}}$ ,  $M_{10}$ . This choice will have implications for proton decay analysis, which I will discuss in the next section.

In light of this establishment of the MSSM doublets, the effective Dirac fermion mass matrices can be written as

$$\begin{aligned}\mathcal{M}_u &= \tilde{h} + r_2 \tilde{f} + r_3 \tilde{g} \\ \mathcal{M}_d &= \frac{r_1}{\tan \beta} (\tilde{h} + \tilde{f} + \tilde{g}) \\ \mathcal{M}_e &= \frac{r_1}{\tan \beta} (\tilde{h} - 3\tilde{f} + c_e \tilde{g}) \\ \mathcal{M}_{\nu_D} &= \tilde{h} - 3r_2 \tilde{f} + c_\nu \tilde{g},\end{aligned}\tag{4.34}$$

where  $1/\tan \beta$  takes  $v_u \rightarrow v_d$  for down-type fields. The couplings with the tildes are given by [43]

$$\begin{aligned}\tilde{h} &\equiv \mathcal{V}_{11} h v_u; \quad \tilde{f} \equiv \frac{\mathcal{U}_{14} f v_u}{r_1 \sqrt{3}}; \quad \tilde{g} \equiv \frac{\mathcal{U}_{12} + \mathcal{U}_{13}/\sqrt{3}}{r_1} g v_u; \\ r_1 &\equiv \frac{\mathcal{U}_{11}}{\mathcal{V}_{11}}; \quad r_2 \equiv r_1 \frac{\mathcal{V}_{15}}{\mathcal{U}_{14}}; \quad r_3 \equiv r_1 \frac{\mathcal{V}_{12} - \mathcal{V}_{13}/\sqrt{3}}{\mathcal{U}_{12} + \mathcal{U}_{13}/\sqrt{3}}; \\ c_e &\equiv \frac{\mathcal{U}_{12} - \mathcal{U}_{13}\sqrt{3}}{\mathcal{U}_{12} + \mathcal{U}_{13}/\sqrt{3}}; \quad c_\nu \equiv r_1 \frac{\mathcal{V}_{12} + \mathcal{V}_{13}\sqrt{3}}{\mathcal{U}_{12} + \mathcal{U}_{13}/\sqrt{3}};\end{aligned}\tag{4.35}$$

where  $\mathcal{U}_{IJ}$ ,  $\mathcal{V}_{IJ}$  are the unitary matrices that diagonalize  $\mathcal{M}_{\mathcal{D}}$ .

The light neutrino mass matrix is given in general by the type-II seesaw mechanism as

$$\mathcal{M}_\nu = f v_L - \mathcal{M}_{\nu_D} (f v_R)^{-1} (\mathcal{M}_{\nu_D})^T;\tag{4.36}$$

I will separately consider the cases of type-I and type-II dominance as outlined previously. Note that the inverse dependence on  $f$  in the type-I term intimately connects the neutrino mass matrix to the charged sector matrices, which makes the model quite predictive. Also note that I will consider only normal mass hierarchy in this analysis.

The matrices  $h$  and  $f$  are real and symmetric, and  $g$  is pure imaginary and anti-symmetric; hence, the Dirac fermion Yukawa couplings are Hermitian in general, and their most general forms can be written as

$$\tilde{h} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & M \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix},$$

$$\tilde{g} = i \begin{pmatrix} 0 & g_{12} & g_{13} \\ -g_{12} & 0 & g_{23} \\ -g_{13} & -g_{23} & 0 \end{pmatrix}. \quad (4.37)$$

$M \equiv h_{33} \sim m_t$  is singled out to stress its dominance over all other elements. The three matrices as written have a total of 15 parameters; taken in combination with ratios  $r_i$  and  $c_\ell$ , the model has a total of 21 parameters. Correspondingly, there are in principle 22 measurable observables, including all masses, mixing angles, and  $CP$  violating phases, associated with the physical fermions, although the three PMNS phases and one neutrino mass have yet to be observed. Therefore one would prefer to have no more than 18 parameters in the model, and generally speaking fewer parameters indicates greater predictability.

Furthermore, as I will discuss in more detail shortly, the dimension-five effective operators that arise in proton decay go like products of Yukawa coupling elements,  $\sim \lambda_{ij} \lambda'_{kl}$  ( $\lambda = h, f, g$ ); therefore, increasing the number of  $\lambda_{ij}$  elements that are small or zero will increase the number of negligible or vanishing contributions to the decay width. This idea was given thorough consideration in [23], and the couplings suggested by the authors are as follows:

$$\tilde{h} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & M \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} \sim 0 & \sim 0 & f_{13} \\ \sim 0 & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix},$$

$$\tilde{g} = i \begin{pmatrix} 0 & g_{12} & g_{13} \\ -g_{12} & 0 & g_{23} \\ -g_{13} & -g_{23} & 0 \end{pmatrix}. \quad (4.38)$$

Note that  $\tilde{h}$  is an explicitly rank-1 matrix, with  $M \sim \mathcal{O}(1)$ ; thus, at leading order, the **10** Higgs  $H \sim m_t$  contributes to the third generation masses and nothing more. This feature has been explored in models demonstrating a discrete flavor symmetry in *e.g.* [44, 45], and may therefore be dynamically motivated. Taking  $f_{12} \sim 0$  is equivalent to a partial diagonalization of  $\tilde{f}$ , which can be done without loss of generality in the

presence of a rank-1  $\tilde{h}$ ; the restriction on  $f_{11}$  is clearly phenomenologically motivated by the smallness of first-generation masses, in the same way the dominance of the parameter  $M$  corresponds to the largeness of third-generation masses. As a result of these assumptions, the above Yukawa texture should give rise to sufficient proton decay lifetimes without the need for the usual extreme cancellations.

It is further preferred for proton decay that  $f_{13}, g_{12} \ll 1$ , although  $f_{13}$  plays a role in setting the size of the reactor neutrino mixing angle  $\theta_{13}$ , so the above restriction may create some tension in the fitting.

In carrying out the numerical minimization, I will allow  $f_{11}$  and  $f_{12}$  to have small but non-vanishing values,  $\mathcal{O}(10^{-4})$ , for the sake of giving accurate first-generation masses without creating tension in other elements. The results of that analysis will be discussed in section 6.1, after I discuss the details of calculating proton decay.

# Chapter 5

## The Details of Proton Decay

In addition to the SM doublets present in each of the GUT Higgs superfields, which contribute to the emergence of  $H_{u,d}$  at the SUSY scale, the heavy fields similarly contain SM-type  $SU(3)$  *color triplets*  $((\mathbf{3}, \mathbf{1}, -\frac{1}{3}) + \text{c.c.})$  in their decompositions. These fields come from the PS components  $\hat{\mathbf{H}}(\mathbf{1}, \mathbf{1}, \mathbf{6})$ ,  $\hat{\Delta}(\mathbf{1}, \mathbf{1}, \mathbf{6}) + \hat{\Delta}_R$ , and  $\hat{\Sigma}(\mathbf{1}, \mathbf{3}, \bar{\mathbf{6}}) + \hat{\Sigma}(\mathbf{1}, \mathbf{1}, \bar{\mathbf{10}})$ . Furthermore, there are two more exotic types of triplets that also lead to  $B$ - or  $L$ -violating vertices:  $(\mathbf{3}, \mathbf{1}, -\frac{4}{3}) + \text{c.c.}$ , which interact with two up-type or two down-type  $SU(2)_L$  singlet fermions, and  $(\mathbf{3}, \mathbf{3}, -\frac{1}{3}) + \text{c.c.}$ , which interact with a pair of  $SU(2)_L$  doublets. The above components have the following couplings to matter superfields in the superpotential:

$$\begin{aligned}
W_{\mathcal{BL}} = & h \left\{ \hat{\mathbf{H}}_{\mathcal{T}} \left( \frac{1}{2} \epsilon_{ab} Q^a Q^b + U^c E^c \right) + \hat{\mathbf{H}}_{\bar{\mathcal{T}}} (\epsilon_{ab} Q^a L^b + U^c D^c) \right\} \\
& + f \left\{ \hat{\Delta}_{\mathcal{T}} \left( \frac{1}{2} \epsilon_{ab} Q^a Q^b - U^c E^c \right) + \hat{\Delta}_{\bar{\mathcal{T}}} (\epsilon_{ab} Q^a L^b - U^c D^c) \right\} + f \sqrt{2} \hat{\Delta}_{\mathcal{T}}^R U^c E^c \\
& + g \sqrt{2} \left\{ \left( -\hat{\Sigma}_{\mathcal{T}}^6 + \hat{\Sigma}_{\mathcal{T}}^{10} \right) U^c E^c + \hat{\Sigma}_{\bar{\mathcal{T}}}^6 U^c D^c + \epsilon_{ab} \hat{\Sigma}_{\mathcal{T}}^{10} Q^a L^b \right\} \\
& + 2f \hat{\Delta}_C D^c E^c + 2g \hat{\Sigma}_C D^c E^c + 2g \hat{\Sigma}_{\bar{C}} U^c U^c \\
& - 4f Q i \sigma_2 \hat{\Delta}_{\bar{Q}} L - 2g Q i \sigma_2 \hat{\Sigma}_Q Q - 4g Q i \sigma_2 \hat{\Sigma}_{\bar{Q}} L,
\end{aligned} \tag{5.1}$$

where I have again suppressed generation and color indices. Note that all of the terms present violate baryon or lepton number. The terms in the final two lines represent the exotic couplings.

Like the doublets, the ordinary color triplets will mix after the GUT-scale breaking to form mass eigenstates; again, this mixing includes triplets contained in the **210** and

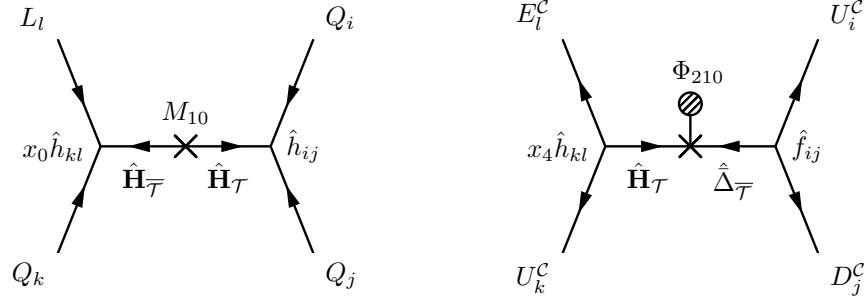


Figure 5.1: Examples of superfield diagrams that lead to proton decay in this model. The hats on the couplings indicate mass basis, and the parameters  $x_i$  contain the triplet mixing information unique to the specific pairing of couplings present in each diagram (see below).

**126** fields not contributing to fermion masses. The resulting  $7 \times 7$  triplet mass matrix  $\mathcal{M}_{\mathcal{T}}$  is diagonalized by  $\mathcal{X}\mathcal{M}_{\mathcal{T}}\mathcal{Y}^T$  to give the eigenstates. The exotic types will mix amongst themselves as well in their own  $2 \times 2$  matrices. These matrices are again fully determined by the heavy vevs and the parameters of the  $SO(10)$  superpotential. Since there is no light triplet analog to  $H_{u,d}$  found in the low-scale particle spectrum, all of the fields can be heavy, although the presence of the same parameters in both the doublet and triplet matrices makes the decoupling of the doublet-triplet behavior a substantial topic itself.

$T$ -channel exchange of conjugate pairs of any of these triplets, through a mass term or interaction with a heavy Higgs field such as **54** or **210**, leads to operators that change two quarks into a quark and a lepton; this is the numerically dominant mechanism through which a proton can decay into a meson and a lepton; corresponding  $s$ -channel decays through the scalar superpartners of these triplets, as well as  $s$ -channel decays through the  $SU(5)$ -like gauge bosons  $X, Y$ , are suppressed by an additional factor of  $1/M_U$  and so are generally negligible in comparison.<sup>1</sup> Figure 5.1 shows Feynman diagrams for two examples of the operators in question.

## 5.1 The Effective Potential

At energies far below the GUT scale, the triplet fields are integrated out, giving four-point effective superfield operators, which give rise in turn to four-fermion operators.

<sup>1</sup>The dominant mode in  $X$ -boson exchange,  $p \rightarrow \pi^0 e^+$ , may be comparable if the relevant threshold corrections are large.

The corresponding effective superpotential is

$$\mathcal{W}_{\mathcal{BL}} = \frac{\epsilon_{\rho\sigma\tau}}{M_{\mathcal{T}}} \left( \widehat{C}_{ijkl}^L Q_i^\rho Q_j^\sigma Q_k^\tau L_l + \widehat{C}_{[ijk]l}^R U_i^{c\rho} D_j^{c\sigma} U_k^{c\tau} E_l^c \right), \quad (5.2)$$

where  $i, j, k, l = 1, 2, 3$  are the generation indices and  $\rho, \sigma, \tau = 1, 2, 3$  are the color indices;  $SU(2)$  doublets are contracted pairwise. This potential has  $\Delta L = 1$  and  $\Delta B = 1$  and so also has  $\Delta(B - L) = 0$ .  $M_{\mathcal{T}}$  is a generic mass for the triplets, which I will take  $\sim M_U$ . Note the anti-symmetrization of  $i, k$  in the  $C_R$  operator; this is the non-vanishing contribution in light of the contraction of the color indices. The analogous anti-symmetry for the  $L$  operator is ambiguous in the current notation, but I will tend to the issue shortly.

The effective operator coefficients  $C_{ijkl}$  are of the form

$$\begin{aligned} C_{ijkl}^R &= x_0 h_{ij} h_{kl} + x_1 f_{ij} f_{kl} + x_2 g_{ij} g_{kl} + x_3 h_{ij} f_{kl} + x_4 f_{ij} h_{kl} + x_5 f_{ij} g_{kl} \\ &\quad + x_6 g_{ij} f_{kl} + x_7 h_{ij} g_{kl} + x_8 g_{ij} h_{kl} + x_9 f_{il} g_{jk} + x_{10} g_{il} g_{jk} \\ C_{ijkl}^L &= x_0 h_{ij} h_{kl} + x_1 f_{ij} f_{kl} - x_3 h_{ij} f_{kl} - x_4 f_{ij} h_{kl} + y_5 f_{ij} g_{kl} + y_7 h_{ij} g_{kl} \\ &\quad + y_9 g_{ik} f_{jl} + y_{10} g_{ik} g_{jl}. \end{aligned} \quad (5.3)$$

The couplings  $h, f, g$  as written correspond to matter fields in the flavor basis and undergo unitary rotations in the change to mass basis, as indicated by the hats on  $\widehat{C}^{L,R}$  in eq. (5.2) above; I will save the details of the change of basis for later in the discussion. The parameters  $x_i, y_i \sim \mathcal{X}_{IJ}, \mathcal{Y}_{IJ}$  are elements of the unitary matrices that diagonalize the triplet mass matrix  $\mathcal{M}_{\mathcal{T}}$ , or the corresponding matrices for the exotic triplets. Note that several identifications have already been made here:  $y_{0,1} = x_{0,1}$  and  $y_{3,4} = -x_{3,4}$ ; looking at eq. (5.1), one can see the would-be parameters  $y_{2,6,8} = 0$ . Also note that  $x_0 \sim M_{10}$  is the **10** mass parameter fixed by the tuning condition for  $M_{\mathcal{D}}$ . The parameters  $x_{9,10}$  and  $y_{9,10}$  correspond to the exotic triplets; the indices of those terms are connected in unique ways as a result of the distinct contractions of fields.

The left-handed term in eq. (5.2) can be further expanded by multiplying out the doublets as

$$\mathcal{W}_{\mathcal{BL}} \ni \frac{\epsilon_{\rho\sigma\tau}}{M_{\mathcal{T}}} \left( \widehat{C}_{[ijk]l}^L U_i^\rho D_j^\sigma U_k^\tau E_l - \widehat{C}_{i[jk]l}^L U_i^\rho D_j^\sigma D_k^\tau \mathcal{N}_l \right), \quad (5.4)$$

where  $\mathcal{N}$  is the left-handed neutrino superfield. Note that the coefficients  $C^L$  are anti-symmetrized in the indices of the like-flavor quarks, again due to the anti-symmetry of color index contraction, as discussed above for  $C^R$ . This anti-symmetry will be crucial

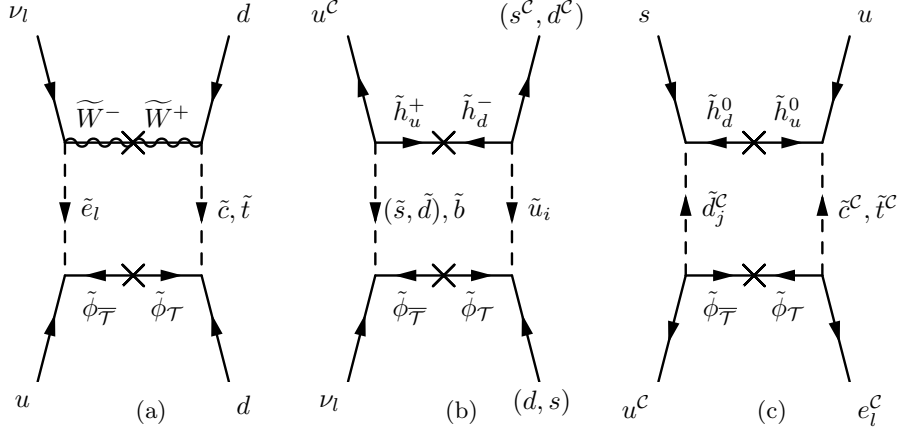


Figure 5.2: Examples of dressed diagrams leading to proton decay in the model.  $\phi = H, \bar{\Delta}, \Sigma$ . Diagram (a) shows a contribution to  $p \rightarrow \pi^+ \bar{\nu}_l$ ; integrating out the triplets gives an effective operator of type  $C^L u d u e$ . Diagram (b) shows a  $C^L u d d \nu$ -type operator contributing to  $K^+ \bar{\nu}_l$ . Diagram (c) shows a  $C^R u^c d^c u^c e^c$ -type operator contributing to  $K^0 e_l^+$ , for  $l = 1, 2$ . Note where more than one field is listed, each choice gives a separate contributing channel, except for the dependent exchange of  $(s \leftrightarrow d)$  in (b).

in restricting the number of contributing channels for decay.

## 5.2 Dressing the Operators

Holomorphism of the superpotential forbids conjugate-mixing mass terms like  $M_\tau \phi_\tau \phi_{\bar{\tau}}$  for  $\phi = H, \bar{\Delta}, \Sigma$  scalar boson components of the triplet superfields; therefore, diagrams of the type in Figure 5.1 can only be realized at leading order through conjugate pairs of *Higgsino* triplet mediators. Thus, in component notation, each vertex will be of the form  $\lambda \tilde{\phi}_\tau q \tilde{q}$  or similar, with  $\lambda = h, f, g$  as appropriate. Therefore, the squarks and sleptons must be “dressed” with gaugino or (SUSY) Higgsino vertices to give  $d = 6$  effective operators of the four-fermion form needed for proton decay. Depending on the sfermions present, diagrams may in principle be dressed with gluinos, Winos, Binors, or Higgsinos. Examples of appropriately-dressed component-field diagrams which give proton decay are shown in Figure 5.2.

In the following subsections, I will discuss the implications for each type of dressing and determine which types will contribute leading factors in the proton decay width. Note that I will give this discussion in terms of  $\tilde{B}, \tilde{W}^0$ , and  $\tilde{h}_{u,d}^{\pm,0}$ , rather than  $\tilde{A}, \tilde{Z}, \tilde{\chi}_i^\pm$ , and  $\tilde{\chi}_i^0$ , because (a) I am assuming a universal mass spectrum for superpartners to satisfy FCNC constraints, meaning the mass and flavor eigenstates coincide for the gauge



bosons, and (b) the mixing of Higgsinos, while not typically negligible, will result in chargino or neutralino masses different from Higgsino mass parameter  $\mu$  by  $\mathcal{O}(1)$  factors as long as gaugino soft masses are relatively small compared to  $M_{\text{SUSY}}$ ; since precise values of such masses are insofar unknown, and since so many of the SUSY and GUT parameter values needed for the decay width calculations are similarly unknown, I will take  $m_{\tilde{h}^\pm} \sim m_{\tilde{h}^0} \sim \mu$  in order to simplify the calculation, especially for computational purposes.

### 5.2.1 Gluino Dressing

Two limitations are readily apparent when considering dressing by gluinos. First, the lepton will have to be a fermion leg in the triplet exchange operator, as in Figure 5.2 (b) or (c), since a slepton cannot be dressed by a gluino. Second, since  $SU(3)_c$  interactions are generation-independent, the gluino can only take  $\tilde{u} \rightarrow u$ ,  $\tilde{s} \rightarrow s$ , etc. The latter may seem a fairly innocuous idea on its own, but consider that proton decay to a kaon or pion will involve operators with one and zero second-generation quarks as external legs, respectively, with all others first-generation. Taking these two points together with the generation-index anti-symmetry of the  $C_{ijkl}$  operators, which implies that  $i \neq k$  for the  $U_i D_j U_k E_l$  operators and  $j \neq k$  for the  $U_i D_j D_k \mathcal{N}_l$  operators, one can see by inspecting a dressed diagram that only diagrams with exactly one each of  $U, D, S$  in the triplet operator may be successfully dressed by the gluino. This constraint implies that gluino dressing can contribute only to  $p \rightarrow K^+ \bar{\nu}$  decay mode; furthermore, the absence of  $UDUE$ -type contributions implies no right-handed channels.

Taking these constraints into account, and thus looking specifically at variants of the  $UDS\mathcal{N}$  operator, there are three independent terms one can write [46], which correspond to the dressed diagrams shown in Figure 5.3:<sup>2</sup>

$$\epsilon_{\rho\sigma\tau} U^\rho D^\sigma S^\tau \mathcal{N}_l \ni \epsilon_{\rho\sigma\tau} \left\{ (u^\rho \nu_l) (\tilde{d}^\sigma \tilde{s}^\tau) + (d^\sigma \nu_l) (\tilde{u}^\rho \tilde{s}^\tau) + (s^\tau \nu_l) (\tilde{u}^\rho \tilde{d}^\sigma) \right\}. \quad (5.5)$$

Applying the gluino dressing to each term gives the following sum of four-fermion effective operators:

$$\xrightarrow{\tilde{g}} \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_s}{4\pi} \right) \left\{ \kappa_1 (u^\rho \nu_l) (d^\sigma s^\tau) + \kappa_2 (d^\sigma \nu_l) (u^\rho s^\tau) + \kappa_3 (s^\tau \nu_l) (u^\rho d^\sigma) \right\}, \quad (5.6)$$

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<sup>2</sup>Each term like “ $(u^\rho \nu_l)$ ” is actually  $(u^\rho)^T C^{-1} \nu_l$ ; the details have been suppressed simply for readability.

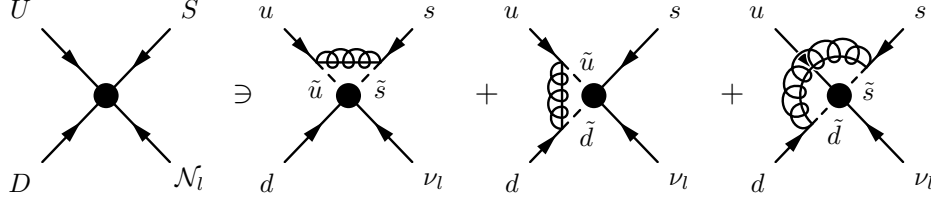


Figure 5.3: Gluino dressings of the  $d = 5$  operator  $M_{\mathcal{T}}^{-1} \hat{C}_{1[12]l}^L UDS\mathcal{N}$  that would contribute to  $p \rightarrow K^+ \bar{\nu}_l$ ; in the limit of universal squark masses, the three diagrams sum to zero by a Fierz identity. NOTE: gluino mass insertions have been omitted from the diagrams for readability.

where the parameters  $\kappa_a$  contain factors from the scalar and gluino propagators in the loop integral. The scalar propagators are different in general; however, recall that I am assuming universality, meaning that all sfermion masses are equal to leading order. In that case, all  $\kappa$ s are equal and can be factored out of the brackets. The sum left inside the brackets is zero by a Fierz identity for fermion contractions [47], and so the contribution from gluino dressing to the  $K^+ \bar{\nu}$  decay mode vanishes under the universal mass assumption.

### 5.2.2 Bino Dressing

As with  $SU(3)_c$ ,  $U(1)_Y$  interactions are also flavor-diagonal; thus, the same constraints apply here as in the gluino case, and possible contributions are to the  $K^+ \bar{\nu}$  mode only.

Looking again at the  $UDS\mathcal{N}$  operator, for terms in which the neutrino is a fermion leg, the argument is analogous to that given for the gluino dressing: the diagrams involved are identical to the three in Figure 5.3 except with  $\tilde{g} \rightarrow \tilde{B}$ ; starting again from expression (5.5) and applying the Bino dressing, one arrives at an expression similar to (5.6) but containing hypercharge coefficients in addition to the  $\kappa_a$ :

$$\begin{aligned} \xrightarrow{\tilde{B}} \quad \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_1}{4\pi} \right) \{ & \kappa_1 Y_d Y_s (u^\rho \nu_l) (d^\sigma s^\tau) + \kappa_2 Y_u Y_s (d^\sigma \nu_l) (u^\rho s^\tau) \\ & + \kappa_3 Y_u Y_d (s^\tau \nu_l) (u^\rho d^\sigma) \}; \end{aligned} \quad (5.7)$$

however,  $u, d, s \in Q_i$  are all left-handed quarks with  $Y = \frac{1}{6}$ , so the hypercharge products factor out, and again the fermion sum vanishes by the Fierz identity.

Because leptons carry hypercharge, there are three additional diagrams one should include in Figure 5.3 if dressing instead by the Bino, namely, those involving the scalar neutrino; these diagrams are shown in Figure 5.4, and the corresponding terms from

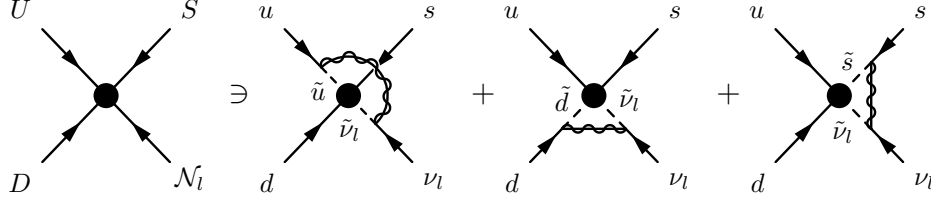


Figure 5.4: Bino dressings of the  $d = 5$  operator  $M_{\mathcal{T}}^{-1} \widehat{C}_{1[12]l}^L UDS\mathcal{N}$  involving a scalar neutrino that would contribute to  $p \rightarrow K^+ \bar{\nu}_l$ ; again, in the limit of universal squark masses, the three diagrams sum to zero by a Fierz identity. NOTE: Bino mass insertions have been omitted from the diagrams for readability.

the triplet operator are

$$\epsilon_{\rho\sigma\tau} U^\rho D^\sigma S^\tau \mathcal{N}_l \ni \epsilon_{\rho\sigma\tau} \left\{ (d^\sigma s^\tau)(\tilde{u}^\rho \tilde{\nu}_l) + (u^\rho s^\tau)(\tilde{d}^\sigma \tilde{\nu}_l) + (u^\rho d^\sigma)(\tilde{s}^\tau \tilde{\nu}_l) \right\}. \quad (5.8)$$

Applying the Bino dressing to each of these terms gives another sum of four-fermion effective operators involving hypercharge:

$$\begin{aligned} \xrightarrow{\tilde{B}} \quad & \kappa \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_1}{4\pi} \right) \left\{ Y_u Y_\nu (d^\sigma s^\tau)(u^\rho \nu_l) + Y_d Y_\nu (u^\rho s^\tau)(d^\sigma \nu_l) \right. \\ & \left. + Y_s Y_\nu (u^\rho d^\sigma)(s^\tau \nu_l) \right\}; \end{aligned} \quad (5.9)$$

this group of terms has a different product of hypercharges from that of (5.7), but it still has a single common product among the three terms, so I can again factor it out, which results in yet another vanishing contribution by the Fierz argument. Hence, the entire Bino dressing contribution to the  $K^+ \bar{\nu}$  mode also vanishes under the universal mass assumption.

### 5.2.3 Wino Dressing

As the flavor-diagonal restrictions of the gluino and Bino also apply to the  $\widetilde{W}^0$  but *not* to the  $\widetilde{W}^\pm$ , the two cases must be considered separately. That said, one additional restriction applicable in both cases is the ability to interact with only left-handed particles; thus there will be no contribution here from the  $R$ -type operators.

**Neutral Wino.** As noted, dressing with the  $\widetilde{W}^0$  is also restricted to  $UDS\mathcal{N}$  contributions to the  $K^+ \bar{\nu}$  mode. The terms to be dressed are the same as those in the Bino case, given by expressions (5.5) and (5.8); however, in applying the dressing, one finds

a kink in the previous argument:

$$\begin{aligned} \xrightarrow{\widetilde{W}^0} & \kappa \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_2}{4\pi} \right) \{ T_d^3 T_s^3 (u^\rho \nu_l) (d^\sigma s^\tau) + T_u^3 T_s^3 (d^\sigma \nu_l) (u^\rho s^\tau) + T_u^3 T_d^3 (s^\tau \nu_l) (u^\rho d^\sigma) \} \\ &= \frac{\kappa \epsilon_{\rho\sigma\tau}}{4} \left( \frac{\alpha_2}{4\pi} \right) \{ (u^\rho \nu_l) (d^\sigma s^\tau) - (d^\sigma \nu_l) (u^\rho s^\tau) - (s^\tau \nu_l) (u^\rho d^\sigma) \}, \end{aligned} \quad (5.10)$$

$$\begin{aligned} \xrightarrow{\widetilde{W}^0} & \kappa \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_2}{4\pi} \right) \{ T_u^3 T_\nu^3 (d^\sigma s^\tau) (u^\rho \nu_l) + T_d^3 T_\nu^3 (u^\rho s^\tau) (d^\sigma \nu_l) + T_s^3 T_\nu^3 (u^\rho d^\sigma) (s^\tau \nu_l) \} \\ &= \frac{\kappa \epsilon_{\rho\sigma\tau}}{4} \left( \frac{\alpha_2}{4\pi} \right) \{ (d^\sigma s^\tau) (u^\rho \nu_l) - (u^\rho s^\tau) (d^\sigma \nu_l) - (u^\rho d^\sigma) (s^\tau \nu_l) \}; \end{aligned} \quad (5.11)$$

the negative weak isospin carried by the down-type fields prevents use of the Fierz identity argument. Thus it seems I have finally found a non-vanishing contribution to proton decay, albeit to only this one mode.

There is something yet to be gained from the Fierz identity in this case: the same zero sum seen in the previous cases tells one that in each expression here, the sum of the two negative terms is equal to the first term; furthermore, note that the final expressions in (5.10) and (5.11) are actually identical. Therefore, I can collect the above contributions into one expression:

$$\begin{aligned} \xrightarrow{\widetilde{W}^0} & 2 \times \frac{\kappa \epsilon_{\rho\sigma\tau}}{4} \left( \frac{\alpha_2}{4\pi} \right) (-2) \{ (u^\rho s^\tau) (d^\sigma \nu_l) + (u^\rho d^\sigma) (s^\tau \nu_l) \} \\ &= - \kappa \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_2}{4\pi} \right) \{ (u^\rho s^\tau) (d^\sigma \nu_l) + (u^\rho d^\sigma) (s^\tau \nu_l) \}. \end{aligned} \quad (5.12)$$

Including the factors from the triplet operator, I can write an operator for the entire neutral Wino contribution to  $K^+ \bar{\nu}$ :

$$\mathcal{O}_{\widetilde{W}^0} = \kappa \epsilon_{\rho\sigma\tau} \left( \frac{\alpha_2}{4\pi} \right) M_{\mathcal{T}}^{-1} \widehat{C}_{1[12]l}^L \{ (u^\rho s^\tau) (d^\sigma \nu_l) + (u^\rho d^\sigma) (s^\tau \nu_l) \}, \quad (5.13)$$

where the sign cancels with that from the  $UDD\mathcal{N}$  term in eq. (5.2). The details of  $\kappa$  will be discussed in the next subsection. Note I could have instead written the above expressions in terms of  $(d^\sigma s^\tau) (u^\rho \nu_l)$  alone; I choose this version simply because the up-up- and down-down-type pairings in the latter expression are not found in Higgsino or charged Wino modes and so are not otherwise used in calculation.

**Charged Wino.** The assumption of universal mass means that the sfermions are simultaneously flavor and mass eigenstates; therefore, the would-be CKM-like unitary

matrix for each is simply the identity,  $U^{\tilde{f}} \sim \mathbb{I}$ . As a result, the unitary matrix present in the fermion-sfermion-Wino couplings is not  $V_{\text{ckm}}$  or  $V_{\text{pmns}}$ , but rather the single unitary matrix corresponding to the fermion rotation. Nonetheless, this rotation allows for the mixing of generations at the dressing vertices, and the limitations found on the neutral current dressings are not applicable. This is quite crucial since it allows for contributions from diagrams with any sfermion propagator not forbidden by the anti-symmetry of the  $C_{ijkl}^L$  operator. Proton decay modes involving neutral kaons or pions, which have  $u\bar{u}$  or  $d\bar{d}$  as external quarks, would be intractable without generation mixing. Such mixing will of course come at the expense of suppression from an off-diagonal element in the pertinent unitary matrix, which will typically be  $\mathcal{O}(10^{-2-3})$ ; hence, one can begin to see an indication of why the  $K^+\bar{\nu}$  mode is so dominant in the full proton decay width.

One additional constraint on charged Wino dressing involves the Wino mass insertion. Unlike the gauginos discussed so far,  $W^\pm$  are the antiparticles of *each other*, rather than either being its own antiparticle. As a result, the Wino mass term is of the form  $M_{\widetilde{W}}\widetilde{W}^+\widetilde{W}^-$ ; in order to involve one  $\widetilde{W}^+$  and one  $\widetilde{W}^-$  in the dressing, the two sfermions involved must be of opposite  $SU(2)$  flavor. As a result, triplet operators of the form  $u\tilde{d}u\tilde{e}$ ,  $\tilde{u}d\tilde{u}e$  (or the RH equivalents),  $u\tilde{d}\tilde{d}\nu$ , and  $\tilde{u}d\tilde{d}\tilde{\nu}$  do not contribute.

Beyond these constraints, the generational freedom of the sfermions leads to numerous contributions to each of the crucial decay modes,  $K^+\bar{\nu}$ ,  $K^0\ell^+$ ,  $\pi^+\bar{\nu}$ , and  $\pi^0\ell^+$ , where  $\ell = e, \mu$ . In particular the  $UDUE$ - and  $UDDN$ -type operators each contribute to *each* mode through multiple channels. A list of all such contributions would likely be overwhelming to the reader no matter how excellent my choices of notation, but one can find the relevant diagrams in Appendix A.

## 5.2.4 Higgsino Dressing

When compared to the others, Higgsino dressing is wildly unconstrained. First, the low-scale Yukawa couplings governing the fermion-sfermion-Higgsino interactions couple a left-handed field to a right-handed one, so clearly the dressing can be applied to both  $C^L$ - and  $C^R$ -type triplet operators. Also, since charged and neutral Higgsinos couple through the same Yukawas, both types of interactions can mix generations, meaning the generation-diagonal constraints on the rest of the neutral-current dressings do not apply to  $\tilde{h}_{u,d}^0$ . The only previously-mentioned restriction that *does* apply is, like the charged Wino, the mass term for the SUSY Higgs couples  $H_u$  to  $H_d$ , so it therefore cannot contribute through the triplet operators with sfermions of like  $SU(2)$  flavor. One

remaining minor restriction is that one will not see the triplet operator  $\tilde{u}du\tilde{e}$  dressed by  $\tilde{h}^\pm$  nor  $u\tilde{d}\tilde{d}\tilde{\nu}$  dressed by  $\tilde{h}^0$  because each would result in an outgoing left-handed anti-neutrino.

One can find cases in the literature (*e.g.* [47]) of Higgsino-dressed contributions being counted as negligible when compared to those from the Wino; this is usually because if one exchanges the  $g_2^2 V_{\text{Cabibbo}}$  found in a typical dominant Wino contribution for a  $y_{ii'}^u y_{kk'}^d \tan \beta$  found in a typical dominant Higgsino contribution, the resulting value will be smaller by at least a factor of  $\mathcal{O}(10)$ . Of course one makes several assumptions in such a comparison:  $\mu \sim M_{\tilde{W}}$  for one, but additionally that (a)  $\tan \beta$  is small or moderate, and (b) the  $C_{ijkl}$  coefficients are usually of roughly the same magnitude for any combination of  $i, j, k, l$  present.

For this analysis, though, neither assumption is valid: I have already mentioned that I will consider large  $\tan \beta$  for maximal applicability; furthermore, due to the rank-1 texture of the  $h$  coupling and the related sparse or hierarchical textures of  $f$  and  $g$  as shown in eq. (4.38), many of the  $C_{ijkl}$  are small or zero, creating large disparities between the values from one contribution to the next. This discrepancy from expectation is further enhanced by the tendency for the unitary matrices  $U^f$ , which give the off-diagonal suppressions at the dressing vertices in this model, to individually deviate from the hierarchical structure of  $V_{\text{ckm}}$ .

To see the extent to which these two properties can lead to surprises in numerical dominance, consider that, for example, I find  $C_{1213}^L \sim C_{3213}^L U_{31}^d$ ; one might expect that  $U_{31}^d \sim V_{ub}$  and  $C_{1213}^L \sim C_{3213}^L$ , so therefore the former term is much larger than the latter, but in fact neither assumption is accurate.

As a result of these model characteristics, I find that the dominant contributions from Higgsino-dressed diagrams are generally comparable to those from Wino-dressed diagrams. This statement further applies to contributions from *right-handed* operators as well. Thus I made no *a priori* assumptions about which of the  $C^L$ - or  $C^R$ -type Higgsino-dressed contributions might be excluded as negligible.

Because both the  $U^c D^c U^c E^c$  operators and the  $\tilde{h}_{u,d}^0$  dressing contribute to all of the pertinent decay modes, the complete list of channels dressed by the Higgsino is considerably more plentiful than that of the Wino and so would be even more overwhelming, but again one can find all of the pertinent diagrams in Appendix A.

### 5.3 Building the Partial Decay Width Formulae

As I discussed above in the Higgsino dressing subsection, the Yukawa texture seen in eq. (4.38) leads to (a) unusually extreme variation in the sizes of the  $C_{ijkl}$  coefficients, depending strongly on the index values present, and (b) textures for the unitary matrices  $U^f$  which deviate substantially from that of  $V_{\text{ckm}}$ . The repercussions of these features clearly extend beyond affecting the relative size of Wino and Higgsino channel contributions. For one, the off-diagonal suppressions  $U_{kk'}^f$  present in most charged Wino diagrams cannot be dependably approximated as  $V_{kk'}^{\text{ckm}}$ ; fortunately, the GUT-scale  $U^f$  are fixed by the fermion fitting, and since the running of such unitary matrices is small, I can simply use them at the  $\widetilde{W}^\pm$  vertices as reasonable approximations to their low-scale counterparts.

Another complication due the Yukawa texture is the disturbance of typically useful assumptions about which channels dominate the calculation. Such assumptions include dominance of Higgsino channels with  $\tilde{t}, \tilde{b}, \tilde{\tau}$  intermediate states or Wino channels  $\propto V_{ii}$  or  $V_{\text{Cabibbo}}$ . In the absence of the general validity of any such simplification, I am compelled to presume that *any* channel might be a non-negligible contribution to decay width.

Thus, I initially treated all possible channels as potentially significant; however, in the interest of saving considerable computational time, I chose an abridged set of contributions to include in my numerical analysis through inspection of tentative calculations, although my threshold for inclusion was quite conservative. It seemed to me that conventional methods of keeping only the most dominant terms for calculation might easily lead to drastically underestimated decay widths, in that if I exclude ten “negligible” terms smaller than leading contributions by a factor of ten, then I have evidently excluded the equivalent of a leading contribution. To fully avoid such folly, I used a cutoff of roughly 1/50 for exclusion, and made cuts on a per-triplet-operator basis, which translates to three or four significant figures of precision in the decay widths.

The Feynman diagrams for all non-vanishing channels of proton decay for the  $K^+\bar{\nu}_l$ ,  $K^0\ell^+$ ,  $\pi^+\bar{\nu}_l$ , and  $\pi^0\ell^+$  modes are catalogued in Appendix A.

Calculation of a proton partial decay width can be broken into three distinct parts. The first part is the evaluation of the “internal”,  $d = 6$  dressed diagrams discussed in the previous subsection; each diagram corresponds to an effective operator of the form  $Xqqq\ell$ , where  $X \sim M_{\mathcal{T}}^{-1} C_{ijkl} \dots$  is a numerical coefficient unique to each decay channel. Note that here each  $q$  is a single quark fermion, not a doublet. The second

part is the evaluation of a hadronic factor that quantifies the conversion of the three external quarks of a dressed diagram—plus one spectator quark—into a proton and a meson. The third and final part is the evaluation of the “external” effective diagram for  $p \rightarrow M\bar{\ell}$  giving the decay width of the proton. I will go through the details of each stage before giving the resulting decay width expressions.

### 5.3.1 Evaluating the Dressed Operators

The evaluation of one such dressed  $d = 6$  box diagram involves calculating the loop integral but no kinematics, because the physical particles carrying real momenta here are the proton and the meson, not the quarks. The loop factor is not divergent and is of the same general form for every channel; furthermore, as the heavy triplets are common to all diagrams and the sfermion masses are assumed to be equal, the only factors in the loop that vary from one channel to the next are the couplings and masses associated with either the Wino or Higgsino. The remaining variation from one diagram to the next depends entirely on the particle flavors, which is apparent in the external fermions and encoded in the  $C_{ijkl}$  coefficients and the unitary matrices involved in rotation to mass basis. Thus, I can write the operator for any pertinent diagram as a generic Wino- or Higgsino coefficient times one of several flavor-specific “sub-operators”; the forms of the general operators are

$$\mathcal{O}_{\widetilde{W}} = \left(\frac{\alpha_2}{4\pi}\right) \left(\frac{1}{M_{\mathcal{T}}}\right) I(M_{\widetilde{W}}, m_{\tilde{q}}) \mathcal{C}_{\widetilde{W}}^A \quad (5.14)$$

and

$$\mathcal{O}_{\tilde{h}} = \left(\frac{1}{16\pi^2}\right) \left(\frac{1}{M_{\mathcal{T}}}\right) I(\mu, m_{\tilde{q}}) \mathcal{C}_{\tilde{h}}^A, \quad (5.15)$$

where<sup>3</sup>

$$I(a, b) = \frac{a}{b^2 - a^2} \left\{ 1 + \frac{a^2}{b^2 - a^2} \log\left(\frac{a}{b}\right) \right\},$$

---

<sup>3</sup>One might notice that this expression for  $I(a, b)$  differs from what is usually given in the literature for analogous proton decay expressions; the discrepancy is due to my inclusion of the universal mass assumption prior to evaluating the loop integral.



and the sub-operators  $\mathcal{C}^{\mathcal{A}}$  are<sup>4</sup>

$$\begin{aligned}
\mathcal{C}_{\widetilde{W}}^I &= \frac{1}{2}(u^T C^{-1} d_j) \widehat{C}_{[ij1]l}^L U_{ii'}^d U_{ll'}^\nu (d_{i'}^T C^{-1} \nu_{l'}) \\
\mathcal{C}_{\widetilde{W}}^{II} &= \frac{1}{2}(u^T C^{-1} e_l) \widehat{C}_{[1jk]l}^L U_{kk'}^d U_{j1}^u (d_{k'}^T C^{-1} u) \\
\mathcal{C}_{\widetilde{W}}^{III} &= -\frac{1}{2}(u^T C^{-1} d_k) \widehat{C}_{1[jk]l}^L U_{j1}^u U_{ll'}^e (u^T C^{-1} e_{l'}) \\
\mathcal{C}_{\widetilde{W}}^N &= -\frac{1}{2}(d_j^T C^{-1} \nu_l) \widehat{C}_{i[jk]l}^L U_{ii'}^d U_{k1}^u (d_{i'}^T C^{-1} u)
\end{aligned} \tag{5.16}$$

for the (charged) Wino,

$$\begin{aligned}
\mathcal{C}_{\widetilde{h}^\pm}^I &= (u^T C^{-1} e_l) \widehat{C}_{[1jk]l}^L y_{kk'}^{d\dagger} y_{j1}^{u\dagger} (d_{k'}^{cT} C^{-1} u^c) \\
\mathcal{C}_{\widetilde{h}^\pm}^{II} &= -(u^T C^{-1} d_k) \widehat{C}_{1[jk]l}^L y_{j1}^{u\dagger} y_{ll'}^{e\dagger} (u^{cT} C^{-1} e_{l'}^c) \\
\mathcal{C}_{\widetilde{h}^\pm}^{III} &= -(d_j^T C^{-1} \nu_l) \widehat{C}_{i[jk]l}^L y_{ii'}^{d\dagger} y_{k1}^{u\dagger} (d_{i'}^{cT} C^{-1} u^c) \\
\mathcal{C}_{\widetilde{h}^\pm}^N &= (u^{cT} C^{-1} d_j^c) \widehat{C}_{[ij1]l}^R y_{ii'}^u y_{ll'}^e (d_{i'}^T C^{-1} \nu_{l'}) \\
\mathcal{C}_{\widetilde{h}^\pm}^V &= (u^{cT} C^{-1} e_l^c) \widehat{C}_{[1jk]l}^R y_{kk'}^u y_{j1}^d (d_{k'}^T C^{-1} u)
\end{aligned} \tag{5.17}$$

for the charged Higgsino, and

$$\begin{aligned}
\mathcal{C}_{\widetilde{h}^0}^I &= -(u^T C^{-1} d_k) \widehat{C}_{[ij1]l}^L y_{i1}^{u\dagger} y_{ll'}^{e\dagger} (u^{cT} C^{-1} e_{l'}^c) \\
\mathcal{C}_{\widetilde{h}^0}^{II} &= -(u^T C^{-1} e_l) \widehat{C}_{[1jk]l}^L y_{kk'}^{d\dagger} y_{j1}^{u\dagger} (d_{k'}^{cT} C^{-1} u^c) \\
\mathcal{C}_{\widetilde{h}^0}^{III} &= (d_j^T C^{-1} \nu_l) \widehat{C}_{i[jk]l}^L y_{i1}^{u\dagger} y_{kk'}^{d\dagger} (u^{cT} C^{-1} d_{k'}^c) \\
\mathcal{C}_{\widetilde{h}^0}^N &= -(u^{cT} C^{-1} d_j^c) \widehat{C}_{[ij1]l}^R y_{i1}^u y_{ll'}^e (u^T C^{-1} e_l) \\
\mathcal{C}_{\widetilde{h}^0}^V &= -(u^{cT} C^{-1} e_l^c) \widehat{C}_{[1jk]l}^R y_{k1}^u y_{jj'}^d (u^T C^{-1} d_{j'})
\end{aligned} \tag{5.18}$$

for the neutral Higgsino, where I have suppressed the color indices everywhere. Again the hats on  $\widehat{C}^{L,R}$  indicate  $\hat{h}, \hat{f}, \hat{g}$  are rotated to the mass basis, which I will discuss in detail shortly. Note that  $UDUE$  and  $UDDN$  operators generally differ by a sign, as do diagrams dressed by  $\tilde{h}_{u,d}^\pm$  and  $\tilde{h}_{u,d}^0$ ; the latter difference arises from the  $SU(2)$  contraction in the SUSY Higgs mass term. These sign differences create the potential for natural cancellation within the absolute squared sums of interfering diagrams, and even for cancellation of entire diagrams with each other in some cases. Also note that the Yukawa couplings are Hermitian in this model, hence the distinction above between

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<sup>4</sup>I do not list the neutral Wino operator again here, but looking back at eq. (5.13), one can see that  $\kappa = I(M_{\widetilde{W}}, m_{\tilde{q}})$ .

$y^f$  and  $y^{f\dagger}$  is not relevant for this work.

I utilized two additional observations to simplify the implementation of the above operators. First, I took values for the superpartner masses such that  $\mu, M_{\widetilde{W}} \ll m_{\widetilde{q}}$ , which implies  $I(a, b) \simeq a/b^2$ . Also, because I am only interested in the combined contribution of the three neutrinos, and because the total contribution is the same whether one sums over flavor states or mass states, I made the replacement  $U_{ll'}^\nu \rightarrow \delta_{ll'}$  for  $\mathcal{C}_{\widetilde{W}}^I$  and took  $l = l' \Rightarrow y_{ll'}^e = m_l^e/v_d$  for  $\mathcal{C}_{\widetilde{h}^\pm}^{IV}$ .

Since the unitary matrices  $U^f$  do not appear in the SM (+ neutrino sector) Lagrangian except in the CKM and PMNS combinations, the non-diagonal SUSY Yukawas  $y^f$  present in the  $\mathcal{C}^A$  are not physically determined. Fortunately in our GUT model full *high-scale* Yukawas are defined by the completely determined fermion sector. Furthermore, it is known that unitary matrices such as the CKM matrix experience only small effects due to SUSY renormalization. Thus, since the low-scale masses are of course known, I can define good approximations to the SUSY Yukawas needed by using the high-scale  $U^f$  to rotate the diagonal mass couplings at the proton scale, divided by the appropriate vevs:

$$y^u = \frac{1}{v_u} U_u (\mathcal{M}_u^{\text{wk}})^D U_u^\dagger,$$

where  $v_u = v \sin \beta$ , or, in component notation,

$$y_{ij}^u = \frac{1}{v_u} \sum_k m_k^u U_{ik}^u U_{jk}^{u*}. \quad (5.19)$$

I can similarly write

$$y_{ij}^d = \frac{1}{v_d} \sum_k m_k^d U_{ik}^d U_{jk}^{d*}$$

$$y_{ij}^e = \frac{1}{v_d} \sum_k m_k^e U_{ik}^e U_{jk}^{e*},$$

where  $v_d = v \cos \beta$ . Mass values used were taken from the current PDG [48]; light masses are run to the 1-GeV scale, top and bottom masses are taken on-shell. Note that since the Yukawa factors always appear in pairs of opposite flavor in the Higgsino operators, and since  $\frac{1}{\sin \beta \cos \beta} \simeq \tan \beta$  for large  $\beta$ , the Higgsino contributions to proton decay are  $\sim \frac{\tan^2 \beta}{v^4}$  for this model.

There are generally two distinct mass-basis rotations possible for each of the  $UDUE^-$ ,  $UDDN^-$ , and  $U^c D^c U^c E^c$ -type triplet operators; the difference between the

two depends on whether the operator is “oriented” (*i.e.*, in the diagram) such that the lepton is a scalar. For a given orientation, a unitary matrix corresponding to the fermionic field at one vertex in the triplet operator will rotate every coupling present in  $C^{L,R}$  pertaining to that vertex; an analogous rotation will happen for the other vertex in the operator. For example, looking at the  $\pi^+ \bar{\nu}_l$  channel in Figure 5.2(a), every coupling  $\lambda_{ij}$  ( $\lambda = h, f, g$ ) from  $C_{ijkl}^L$  present at the  $\tilde{\phi}_{\mathcal{T}}$  vertex will be rotated by some form of  $U^d$ ; similarly all  $\lambda'_{kl}$  present at the  $\tilde{\phi}_{\overline{\mathcal{T}}}$  vertex will be rotated by some  $U^u$ . The down quark field shown is a mass eigenstate quark resulting from the unitary rotation, which one can interpret as a linear combination of flavor eigenstates:  $d_j = U_{jm}^d d'_m$ , with  $j = 1$ ; applying the same thinking to the up quark, one also has  $u_k^T = u_p'^T U_{pk}^{uT}$ , with  $k = 1$ . To work out the details of the rotations, I start with the  $d = 5$  operator written in terms of flavor states<sup>5</sup>,  $\sum_a x_a (\tilde{u}_i \lambda_{im}^a d'_m) (u_p'^T \lambda_{pl}'^a \tilde{e}_l)$ , where I have expanded  $C_{impl}^L$  in terms of its component couplings and chosen the indices with the malice of forethought; now I can write

$$\begin{aligned} & \sum_a x_a (\tilde{u}_i^T C^{-1} \lambda_{im}^a d'_m) (u_p'^T \lambda_{pl}'^a C^{-1} \tilde{e}_l) \\ &= \sum_a x_a (\tilde{u}_i^T C^{-1} \underbrace{\lambda_{im}^a U_{mj}^{d\dagger}}_{\equiv \hat{\lambda}_{ij}^a} \underbrace{U_{jn}^d d'_n}_{d_j} (\underbrace{u_p'^T U_{pk}^{uT}}_{u_k^T} \underbrace{U_{kq}^{u*} \lambda_{ql}'^a}_{\equiv \hat{\lambda}_{kl}'^a} C^{-1} \tilde{e}_l). \end{aligned}$$

Using the new definitions for  $\hat{\lambda}$ , one can see that the rotated coefficient  $\hat{C}^L$  corresponding to the expression in eq. (5.3) has become

$$\begin{aligned} \hat{C}_{ijkl}^L &= x_0 \hat{h}_{ij} \hat{h}_{kl} + x_1 \hat{f}_{ij} \hat{f}_{kl} - x_3 \hat{h}_{ij} \hat{f}_{kl} + \dots \\ &= x_0 (h U_d^\dagger)_{ij} (U_u^* h)_{kl} + x_1 (f U_d^\dagger)_{ij} (U_u^* f)_{kl} - x_3 (h U_d^\dagger)_{ij} (U_u^* f)_{kl} + \dots \end{aligned} \quad (5.20)$$

Note that this version of  $\hat{C}^L$  is only valid for  $\tilde{u}_i d_j u_k \tilde{e}_l$ -type operators, with this particular orientation in the diagram; there is an analogous pair of rotations for  $u \tilde{d} \tilde{u} e$ , as well as two each for  $UDD\mathcal{N}$  and  $U^c D^c U^c E^c$ , giving a total of six possible schemes.

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<sup>5</sup>Recall the scalars are both mass and flavor eigenstates under the universal mass assumption. Also note “ $\lambda$ ” is again my name for the second generic coupling, and the prime has nothing to do with basis; I will continue to use hats to indicate rotated couplings.

### 5.3.2 From Quarks to Hadrons

As mentioned above, the composite hadrons  $p$  and  $K, \pi$  (in addition to the lepton) carry physical momenta in the proton decay process, *not* the “external”, “physical” quarks seen in the dressed operators above. Therefore one is in need of calculating a factor like  $\langle M | (qq)q | p \rangle$ , where  $M = K, \pi$  is the final meson state. More explicitly these objects will look like

$$\begin{aligned} & \langle K^+ | \epsilon_{\rho\sigma\tau} (u^\tau s^\sigma)_L d_L^\rho | p \rangle \\ & \langle K^0 | \epsilon_{\rho\sigma\tau} (u^\rho s^\tau)_R u_L^\sigma | p \rangle \\ & \langle \pi^0 | \epsilon_{\rho\sigma\tau} (u^\sigma d^\tau)_L u_R^\rho | p \rangle \\ & \vdots \end{aligned}$$

Such matrix elements are calculated using either chiral Lagrangian methods or a three-point function (for  $M$ ,  $p$ , and the  $(qq)q$  operator) on the lattice; in either case, the result is determined in part by a scaling parameter  $\beta_H$  defined by  $\langle 0 | (qq)q | p(s) \rangle = \beta_H P_L u_p(s)$ , where  $P_L$  is the left-chiral projection matrix and  $u_p(s)$  is the Dirac spinor for an incoming proton of spin  $s$ . In principle  $\beta_H$  is not necessarily the same for cases where the quarks have different chiralities, but the values usually differ only in sign, which is irrelevant when the entire factor is squared in the decay width expression.

While lattice methods have advanced significantly since the early years of SUSY GUT theory, there is still a substantial amount of uncertainty present in the calculation of both  $\beta_H$  and the matrix element factors; some groups have even obtained contradictory results when applying the two methods in the same work [49]. Some more recent works (*e.g.* [50]) using more advanced statistics and larger lattices seem to be converging on trustworthy answers, but it is still normal to see results vary by factors of (1/2 - 5) for a single decay mode from one method to the next, where the values for the matrix elements themselves are  $\mathcal{O}(10) \times \beta_H$ . Thus I will simply take the admittedly favorable approach of using  $\langle M | (qq)q | p(s) \rangle \sim \beta_H P u_p$  for all modes.

It is not uncommon to see values as low as  $\beta_H = 0.003$  used in other works calculating proton decay [51], but while calculated values have indeed varied as much as (0.003 - 0.65) over the years [50], the value is now most commonly found in the range (0.006 - 0.03) [52], with a tendency to prefer  $\beta_H \sim 0.015$ , as seen in [50]. Again, I will take a slightly optimistic approach and use  $\beta_H = 0.008$ .

### 5.3.3 The $p \rightarrow M\bar{\ell}$ Effective Diagram and the Decay Width of the Proton

Ultimately it is a deceptively simple two-body decay that I am calculating, as shown in Figure 5.5. The corresponding decay width can be determined by the usual phase-space integral expression:

$$\Gamma = \frac{1}{2M_p} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_M} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\ell} (2\pi)^4 \delta^4(p_p - p_M - p_\ell) \frac{1}{2} \sum_s |\mathcal{M}|^2 \quad (5.21)$$

where in this case

$$\frac{1}{2} \sum_s |\mathcal{M}|^2 = \frac{1}{2} \beta_H^2 (A_L A_S)^2 (|\mathcal{O}_{\widetilde{W}}|^2 + |\mathcal{O}_{\widetilde{h}}|^2) \sum_{s,s'} |v_\ell^T(p_\ell, s) C^{-1} u_p(p_p, s')|^2. \quad (5.22)$$

The factors  $A_L$  and  $A_S$  arise due to the renormalization of the  $d = 6$  dressed operators, from  $M_p$  to  $M_{\text{SUSY}}$  and  $M_{\text{SUSY}}$  to  $M_U$ , respectively; their values have been calculated in the literature as  $A_L = 0.4$  and  $A_S = 0.9\text{--}1.0$  [53]. The spinor factor can be evaluated with the usual trace methods; in the rest frame of the proton, where  $-\mathbf{p}_M = \mathbf{p}_\ell \equiv \mathbf{p}$ , and utilizing  $m_\ell^2 \ll |\mathbf{p}|^2$  (which is only marginally valid for the muon but clearly so otherwise), the decay width expression simplifies to

$$\Gamma = \frac{1}{4\pi} \beta_H^2 (A_L A_S)^2 (|\mathcal{O}_{\widetilde{W}}|^2 + |\mathcal{O}_{\widetilde{h}}|^2) p, \quad (5.23)$$

where

$$p \equiv |\mathbf{p}| \simeq \frac{M_p}{2} \left( 1 - \frac{m_M^2}{M_p^2} \right). \quad (5.24)$$

Note that  $p \sim M_p/2$  for pion modes, but that value is reduced by a factor of  $\sim 25\%$  for kaon modes.

I now have all the pieces needed to write the working formulae for the partial decay widths of the proton. Let me first define  $C^A$  as extended forms of the  $C_{ijkl}$  by

$$\begin{aligned} \mathcal{C}_{\widetilde{W}}^A &= C_{\widetilde{W}}^A(qq)(q\ell) \\ \mathcal{C}_{\widetilde{h}^\pm}^A &= C_{\widetilde{h}^\pm}^A(qq)(q\ell) \\ \mathcal{C}_{\widetilde{h}^0}^A &= C_{\widetilde{h}^0}^A(qq)(q\ell), \end{aligned} \quad (5.25)$$

so that these coefficients contain the  $U^f$  or  $y^f$  factors as well as the  $C^{L,R}$  of the  $\mathcal{C}^A$

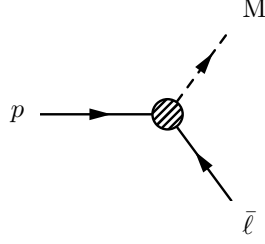


Figure 5.5: Proton decay to a meson and an anti-lepton; the effective operator vertex contains hadronic and renormalization factors as well as the sum of all  $d = 6$  dressed operators contributing to the mode.

operators in (5.16)-(5.18). Now I can easily translate an operator expression like

$$\mathcal{O}_{\widetilde{W}}(K^+\bar{\nu}) \simeq \left(\frac{\alpha_2}{4\pi}\right) \frac{1}{M_{\mathcal{T}}} \left(\frac{M_{\widetilde{W}}}{m_q^2}\right) \{\mathcal{C}_{\widetilde{W}}^I + \mathcal{C}_{\widetilde{W}}^{IV}\} \quad (5.26)$$

into a partial decay width statement,

$$\Gamma_{\widetilde{W}}(p \rightarrow K^+\bar{\nu}) \simeq \frac{1}{4\pi} \left(\frac{\alpha_2}{4\pi}\right)^2 \frac{1}{M_{\mathcal{T}}^2} \left(\frac{M_{\widetilde{W}}}{m_q^2}\right)^2 \beta_H^2 (A_L A_S)^2 \text{p} |C_{\widetilde{W}}^I + C_{\widetilde{W}}^{IV}|^2, \quad (5.27)$$

without losing either information or readability. Note though there is still a “black-box” nature to the  $\mathcal{C}^{\mathcal{A}}$  (it was there in the  $\mathcal{C}^{\mathcal{A}}$  operators as well), in that without specifying the generation indices of the external  $d_{j,i'}$  quarks, the sums in eqs. (5.26) and (5.27) could just as easily apply to  $\pi^+\bar{\nu}$ . Furthermore, there are at least several channels present in each  $\mathcal{C}^{\mathcal{A}}$  operator that contribute to any one mode, which are determined uniquely by the generations of the internal sfermions in addition to those of the external quarks.<sup>6</sup> If the reader wishes to examine the decay widths at the full level of detail, he or she should utilize these expressions along with the operators in eqs. (5.16)-(5.18) and the diagrams in Appendix A.

All remaining limitations aside, I can now present relatively compact and intelligible expressions for the Wino- and Higgsino-dressed partial decay widths of the proton

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<sup>6</sup>Indeed I could have defined the coefficients with six indices:  $C_{ijklmn}^{\mathcal{A}}$ , thereby creating a means of alleviating all degeneracy, but I do not expect such information-dense objects to be so enlightening to readers, especially since for most modes, at least the Higgsino-dressed expression would devolve into an entire pageful of terms corresponding to the individual channels.

for generic mode  $p \rightarrow M\bar{\ell}$ :

$$\Gamma_{\widetilde{W}}(p \rightarrow M\bar{\ell}) \simeq \frac{1}{4\pi} \left( \frac{\alpha_2}{4\pi} \right)^2 \frac{1}{M_{\mathcal{T}}^2} \left( \frac{M_{\widetilde{W}}}{m_{\tilde{q}}^2} \right)^2 \beta_H^2 (A_L A_S)^2 \text{p} \left| \sum_{\mathcal{A} \in M\bar{\ell}} C_{\widetilde{W}}^{\mathcal{A}} \right|^2 \quad (5.28)$$

$$\Gamma_{\tilde{h}}(p \rightarrow M\bar{\ell}) \simeq \frac{1}{4\pi} \left( \frac{1}{16\pi^2} \right)^2 \frac{1}{M_{\mathcal{T}}^2} \left( \frac{\mu}{m_{\tilde{q}}^2} \right)^2 \beta_H^2 (A_L A_S)^2 \text{p} \left| \sum_{\mathcal{A} \in M\bar{\ell}} C_{\tilde{h}}^{\mathcal{A}} \right|^2. \quad (5.29)$$

For the numerical analysis, I used the generic values  $M_{\mathcal{T}} = 2 \times 10^{16} \text{ GeV}$ ,  $M_{\widetilde{W}} = \mu = 100 \text{ GeV}$ , and  $m_{\tilde{q}} = 3 \text{ TeV}$ . Also, let me repeat here that because of the two SUSY Yukawa coupling factors in the  $C_h^{\mathcal{A}}$ , which always come in opposite flavor,

$$\Gamma_{\tilde{h}} \propto \left( \frac{1}{v^2 \sin \beta \cos \beta} \right)^2 \sim \frac{\tan^2 \beta}{v^4}.$$

Before moving on to the fermion sector fit results, let me remark that because the Higgsinos vertices change the chiralities of the outgoing fermions, there can be no interference between Wino- and Higgsino-dressed diagrams, as implied by the notation in eq. (5.23); however, since diagrams for the right-handed  $C^R$  operators have outgoing *left-handed* fermions by the same Higgsino mechanism, diagrams for  $C^R$ - and  $C^L$ -type operators with the same external particles of matching chiralities *do* interfere with each other, and so all such contributions to a given mode do in fact go into the same absolute-squared sum factor, as suggested by eq. (5.29).

# Chapter 6

## Results of the Analysis

### 6.1 Fitting the Fermion Mass Matrices

Diagonalizing the mass matrices given in eq.(4.34), with the Yukawa textures shown in (4.38), gives the GUT-scale fermion masses and mixing angles for a given set of values for the mass matrix parameters  $h_{ij}$ ,  $f_{ij}$ ,  $r_i$ , etc. In order to find the best fit to the experimental data, I used the `Minuit` tool library for Python [54, 55] to minimize the sum of chi-squares for the mass-squared differences  $\Delta m_{21}^2$  (aka  $\Delta m_{\odot}^2$ ) and  $\Delta m_{32}^2$  (aka  $\Delta m_{\text{atm}}^2$ ) and the PMNS mixing angles in the neutrino sector as well as the mass eigenvalues and CKM mixing angles in the charged-fermion sector. Type-I and type-II seesaw neutrino masses were each fit independently, so I report the results for each separately.

Note that throughout the analysis, I have taken  $v_u = 117.8 \text{ GeV}$ , which is calculated with  $\tan \beta = 55$  and for  $v$  run to the GUT scale [56]. The corresponding value for the down-type vev is  $v_d = 2.26 \text{ GeV}$ .

Threshold corrections at the SUSY scale are  $\propto \tan \beta$ , and so should be large in this analysis [57]. The most substantial correction is to the bottom quark mass, which is dominated by gluino and chargino loop contributions; this correction also induces changes to the CKM matrix elements involving the third generation. The explicit forms of these corrections can be seen in a previous work on a related model [45]. Additionally, smaller off-diagonal threshold corrections to the third generation parts of  $\mathcal{M}_d$  result in small corrections to the down and strange masses as well as further adjustments to the CKM elements. All such corrections can be parametrized in the



model by

$$\mathcal{M}'_d = \mathcal{M}_d + \frac{r_1}{\tan \beta} \begin{pmatrix} 0 & 0 & \delta V_{ub} \\ 0 & 0 & \delta V_{cb} \\ \delta V_{ub} & \delta V_{cb} & \delta m_b \end{pmatrix}, \quad (6.1)$$

where  $\mathcal{M}_d$  is given by eq. (4.34). If I simply take this augmented form for  $\mathcal{M}_d$  as part of the model input, the  $\delta$  parameters are fixed by the mass matrix fitting, which results in implied constraints on certain SUSY parameters and the mass values that depend on them, namely, the Higgs and the light stop and sbottom masses. This entire prescription and its implications were considered in detail in [45], and in comparing to that work, one can see that for large  $\tan \beta$  and relatively small threshold corrections, the resulting constraints on the Higgs and squark masses are less interesting, so I will not consider them in more detail for this analysis.

### 6.1.1 Fit Results for Type II Seesaw

If one breaks  $SO(10)$  and  $B - L$  together at  $v_R \gtrsim 10^{17}$  GeV, and sets the vev  $v_L \sim 1$  eV through a tuning of the  $SU(5)$  **15** mass term for  $\bar{\Delta}_L$ , then the  $v_L$  term in eq. (4.36) dominates over the type-I contribution by 2-4 orders of magnitude in the neutrino mass matrix; therefore eq. (4.36) reduces to

$$\mathcal{M}_\nu \simeq v_L f \quad (6.2)$$

Using this prescription, I find a fairly large parameter space for which the sum of chi-squares is quite low, although some of the output values, such as  $\theta_{13}$  and the down and bottom masses, are quite sensitive to the variation in the minima. This is problematic for  $\theta_{13}$  especially, since it is known to high experimental precision [58]. Tables 6.1 and 6.2 display the properties of one of the more favorable fits; Table 6.1 gives the values for the adjusted model input parameters, and Table 6.2 gives the corresponding output values for the fermion parameters, with experimentally measured values included for comparison. Note that the down quark mass is seemingly a bit low, which seems to be a general feature in this model, but I will discuss in the next section why this is not a problem. The precise value of  $v_L$  for this fit is 1.316 eV, which sets the overall neutrino mass scale at  $m_3 \sim 0.05$  eV.

In order to calculate the  $C_{ijkl}$  proton decay coefficients, as well as for use in

$M$ (GeV)	106.6	$r_1/\tan\beta$	0.014601
$f_{11}$ (GeV)	-0.045564	$r_2$	0.0090315
$f_{12}$ (GeV)	0.048871	$r_3$	1.154
$f_{13}$ (GeV)	-0.59148	$c_e$	-2.5342
$f_{22}$ (GeV)	-2.06035	$c_\nu$	n/a
$f_{23}$ (GeV)	-1.4013	$\delta m_b$ (GeV)	-22.740
$f_{33}$ (GeV)	-1.40644	$\delta V_{cb}$ (GeV)	1.2237
$g_{12}$ (GeV)	0.018797	$\delta V_{ub}$ (GeV)	4.2783
$g_{13}$ (GeV)	-0.92510		
$g_{23}$ (GeV)	-3.8353		

Table 6.1: Best fit values for the model parameters at the GUT scale with type-II seesaw. Note that  $c_\nu$ , which appears in the Dirac neutrino mass contribution to the type-I term, is not relevant for type-II.

the neutrino mass matrix (4.36), I needed to determine the “raw” Yukawa couplings,  $h, f, g$ , from the dimensionful couplings,  $\tilde{h}, \tilde{f}, \tilde{g}$ , of the mass matrices given in eq. (4.34), which are obtained directly from the fit; to do so I need to extract the absorbed vev  $v_u$  and doublet mixing parameters  $f(\mathcal{U}_{IJ}, \mathcal{V}_{IJ})$  discussed in section 4.4. There is some freedom in the values of those mixing elements from the viewpoint of this predominantly phenomenological analysis, but they are constrained by both unitarity and the ratios  $r_i$  and  $c_\ell$ , which have been fixed by the fermion fit. Again, see [43] for details, or see [45] for an example of such a calculation. The resulting dimensionless couplings corresponding to this type-II fit are

$$\begin{aligned}
h &= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1.207 \end{pmatrix}; & f &= \begin{pmatrix} -0.00053748 & 0.00057649 & -0.0069772 \\ 0.00057649 & -0.024304 & -0.016530 \\ -0.0069772 & -0.016530 & -0.0165906 \end{pmatrix} \\
g &= i \begin{pmatrix} 0 & 0.00033485 & -0.016480 \\ -0.00033485 & 0 & -0.0683214 \\ 0.016480 & 0.0683214 & 0 \end{pmatrix}
\end{aligned} \tag{6.3}$$

Note that in addition to  $f_{11} \sim f_{12} \sim 0$ , this fit satisfies  $g_{12}, f_{13} \ll 1$  as is desired for proton decay.

	best fit	exp value		best fit	exp value
$m_u$ (MeV)	0.7172	$0.72^{+0.12}_{-0.15}$	$V_{us}$	0.2245	$0.2243 \pm 0.0016$
$m_c$ (MeV)	213.8	$210.5^{+15.1}_{-21.2}$	$V_{ub}$	0.00326	$0.0032 \pm 0.0005$
$m_t$ (GeV)	106.8	$95^{+69}_{-21}$	$V_{cb}$	0.0349	$0.0351 \pm 0.0013$
$m_d$ (MeV)	0.8827	$1.5^{+0.4}_{-0.2}$	$J \times 10^{-5}$	2.38	$2.2 \pm 0.6$
$m_s$ (MeV)	34.04	$29.8^{+4.18}_{-4.5}$	$\Delta m_{21}^2 / \Delta m_{32}^2$	0.03065	$0.0309 \pm 0.0015$
$m_b$ (GeV)	1.209	$1.42^{+0.48}_{-0.19}$	$\theta_{13}$ ( $^\circ$ )	9.057	$8.88 \pm 0.385$
$m_e$ (MeV)	0.3565	$0.3565^{+0.0002}_{-0.001}$	$\theta_{12}$ ( $^\circ$ )	33.01	$33.5 \pm 0.8$
$m_\mu$ (MeV)	75.297	$75.29^{+0.05}_{-0.19}$	$\theta_{23}$ ( $^\circ$ )	47.70	$44.1 \pm 3.06$
$m_\tau$ (GeV)	1.635	$1.63^{+0.04}_{-0.03}$	$\delta_{CP}$ ( $^\circ$ )	-7.506	
			$\sum \chi^2$	6.0	

Table 6.2: Best fit values for the charged fermion masses, solar-to-atmospheric mass squared ratio, and CKM and PMNS mixing parameters for the fit with Type-II seesaw. The  $1\sigma$  experimental values are also shown for comparison [56], [48], where masses and mixings are extrapolated to the GUT scale using the MSSM RGEs. Note that the fit values for the bottom quark mass and the CKM mixing parameters involving the third generation shown here include the SUSY-threshold corrections

### 6.1.2 Fit Results for Type I Seesaw

If one instead takes  $v_R \lesssim 10^{16}$  GeV and  $v_L \sim 1$  meV, then the type-I contribution is dominant over the type-II contribution, and eq. (4.36) becomes

$$\mathcal{M}_\nu \simeq -\mathcal{M}_{\nu_D} (v_R f)^{-1} (\mathcal{M}_{\nu_D})^T, \quad (6.4)$$

In this case, initial searches again showed that certain output parameters were quite sensitive to the input and were often in contention with each other or with the *de facto* upper bounds on the  $f_{ij}$  needed for proton decay. In the first cluster of minima found by the fitting, the output values for one or more of charm mass, bottom mass, or  $\theta_{23}$  was much too small; furthermore, those results came with odd, large tunings of certain input parameters, such as  $c_{e,\nu} \sim \mathcal{O}(100)$  or  $\delta m_b > 40$  GeV. The addition of a small type-II correction to the neutrino matrix led me to a new swath of parameter space, and ultimately I found a new cluster of minima that did not require the correction. Table 6.3 gives the values for the adjusted model input parameters for one such pure type-I fit, and Table 6.4 gives the corresponding output values for the fermion parameters. Fits in this swath of parameter space still have  $c_\nu \sim 50$  and  $\delta m_b \sim 25$  GeV, but this value for  $c_\nu$ , while slightly strange, can be accommodated by the freedom in the

$M$ (GeV)	76.10	$r_1/\tan\beta$	0.024701
$f_{11}$ (GeV)	0.010130	$r_2$	0.24414
$f_{12}$ (GeV)	-0.089576	$r_3$	0.00600
$f_{13}$ (GeV)	0.93973	$c_e$	-3.3279
$f_{22}$ (GeV)	0.8659	$c_\nu$	45.218
$f_{23}$ (GeV)	1.4884	$\delta m_b$ (GeV)	-28.000
$f_{33}$ (GeV)	3.5495	$\delta V_{cb}$ (GeV)	-0.84394
$g_{12}$ (GeV)	0.20048	$\delta V_{ub}$ (GeV)	0.51486
$g_{13}$ (GeV)	0.05352		
$g_{23}$ (GeV)	0.35153		

Table 6.3: Best fit values for the model parameters at the GUT scale with type-I seesaw.

doublet mixing parameters, and such a value for the largest SUSY threshold correction is actually quite moderate for large  $\tan\beta$ . The precise value for the  $\bar{\Delta}_R$  vev in this fit is  $v_R = 1.21 \times 10^{15}$  GeV.

Note also that the top and strange masses are quite a bit lower than in the type-II fit; however, note I have also quoted different experimental values with which agreement is maintained. The differences here come from an update to the work in [56] in determining two-loop MSSM RGEs for fermion masses. The update [59] reports notably lower masses for all the quarks at  $\tan\beta = 55$  and  $\mu = 2.0 \times 10^{16}$  GeV, especially for the up, down, strange, and top masses, due to updates in initial values and methodology. Hence, one should not give the specific values too much weight in such a fit, and I do not consider the reported differences to be significant. This same thinking applies for the type-II down mass value in Table 6.2.

Again I need to determine the raw Yukawa couplings for proton decay analysis. The resulting couplings corresponding to this type-I fit are

$$\begin{aligned}
h &= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1.6152 \end{pmatrix} & f &= \begin{pmatrix} 0.0001623 & -0.00143525 & 0.01505699 \\ -0.00143525 & 0.01387415 & 0.02384774 \\ 0.01505699 & 0.02384774 & 0.05687217 \end{pmatrix} \\
g &= i \begin{pmatrix} 0 & 0.0068081 & 0.0018175 \\ -0.0068081 & 0 & 0.0119376 \\ -0.0018175 & -0.0119376 & 0 \end{pmatrix} & & (6.5)
\end{aligned}$$

Here, one still finds  $f_{11} \sim 0$ , but each of  $f_{12}$ ,  $f_{13}$ , and  $g_{12}$  is larger by an order of

	best fit	exp value		best fit	exp value
$m_u$ (MeV)	0.72155	$0.72^{+0.12}_{-0.15}$	$V_{us}$	0.2240	$0.2243 \pm 0.0016$
$m_c$ (MeV)	212.2	$210.5^{+15.1}_{-21.2}$	$V_{ub}$	0.00310	$0.0032 \pm 0.0005$
$m_t$ (GeV)	76.97	$80.45^{+2.9*}_{-2.6}$	$V_{cb}$	0.0352	$0.0351 \pm 0.0013$
$m_d$ (MeV)	1.189	$0.930 \pm 0.38^*$	$J \times 10^{-5}$	2.230	$2.2 \pm 0.6$
$m_s$ (MeV)	20.81	$17.6^{+4.9*}_{-4.7}$	$\Delta m_{21}^2 / \Delta m_{32}^2$	0.0309	$0.0309 \pm 0.0015$
$m_b$ (GeV)	1.278	$1.24 \pm 0.06^*$	$\theta_{13}$ ( $^\circ$ )	8.828	$8.88 \pm 0.385$
$m_e$ (MeV)	0.3565	$0.3565^{+0.0002}_{-0.001}$	$\theta_{12}$ ( $^\circ$ )	33.58	$33.5 \pm 0.8$
$m_\mu$ (MeV)	75.29	$75.29^{+0.05}_{-0.19}$	$\theta_{23}$ ( $^\circ$ )	41.76	$44.1 \pm 3.06$
$m_\tau$ (GeV)	1.627	$1.63^{+0.04}_{-0.03}$	$\delta_{CP}$ ( $^\circ$ )	-46.3	
			$\sum \chi^2$	1.75	

Table 6.4: Best fit values for the charged fermion masses, solar-to-atmospheric mass squared ratio, and CKM and PMNS mixing parameters for the fit with Type-I seesaw. The  $1\sigma$  experimental values are shown [56] (\* - from [59] instead), [48]; masses and mixings are extrapolated to the GUT scale using the MSSM RGEs. Note that again that pertinent fit values include threshold corrections.

magnitude than in the type-II case, which is thought to be unfavorable for proton decay. At the same time,  $g_{13}$  and  $g_{23}$  are smaller by an order of magnitude, so it is not clear that the net benefit lost is substantial. In the end, a different distinction will give way to success for this type-I fit; I will discuss those details in the next section.

## 6.2 Results of Calculating Proton Partial Lifetimes

In order to give an actual number for any decay width, in addition to choosing representative values for the triplet, sfermion, and Wino or Higgsino masses, I also need values for the  $x_i$  and  $y_i$  triplet mixing parameters in order to calculate the  $C_{ijkl}$  values. Recall that the **10** mass parameter  $x_0$  must be fixed at  $\mathcal{O}(1)$  to allow the SUSY Higgs fields to be light; the remaining mixing parameters are functions of many undetermined GUT-scale masses and couplings found in the full superpotential for the heavy Higgs fields, the details of which can be seen in [42]. There are nearly as many of those GUT parameters as there are independent  $x$ s and  $y$ s, so it is not unreasonable to simply treat the latter as free parameters.

Ideally, one would find that the width for any particular mode would be essentially independent of those parameter values, *i.e.*, that for arbitrary choices  $0 < |x_i|, |y_i| < 1$ , devoid of unlucky relationships leading to severe enhancements, all mode lifetimes would

decay mode	$\tau$ exp lower limit (yrs)
$p \rightarrow K^+ \bar{\nu}$	$6.0 \times 10^{33}$
$p \rightarrow K^0 e^+$	$1.0 \times 10^{33}$
$p \rightarrow K^0 \mu^+$	$1.3 \times 10^{33}$
$p \rightarrow \pi^+ \bar{\nu}$	$2.7 \times 10^{32}$
$p \rightarrow \pi^0 e^+$	$1.3 \times 10^{34}$
$p \rightarrow \pi^0 \mu^+$	$1.0 \times 10^{34}$

Table 6.5: Experimentally determined lower limits [60] on the partial lifetimes of dominant proton decay modes considered in this work.

be comfortably clear of the experimentally determined lower limits, given in Table 6.5. The reality is quite bleak in comparison. For a typical GUT model, if the proton decay lifetimes can be satisfied at all, one is required to choose  $x$  and  $y$  values very carefully such that either individual  $C_{ijkl}$  or  $\left| \sum C^A \right|$  are small through cancellations among terms. These tunings may need to be several orders of magnitude in size (*e.g.*,  $C^A = -C^B + \mathcal{O}(10^{-3})$ ), and many such relationships may be needed.

The Yukawa textures shown in eq. (4.38) are intended to naturally suppress the values of some crucial  $C_{ijkl}$  values so that the need for such extreme tuning is alleviated. In order to test the ansatz, I “simply” needed to find a set of values for the mixing parameters yielding partial decay widths that satisfy the experimental constraints; the difficulty in determining those values inversely corresponds to success of the ansatz. If the ansatz does indeed work optimally, I should be able to choose arbitrary  $x_i$  and  $y_i$  values as suggested above. Realistically though, the authors of [23] and I expected some searching for a valid region of parameter space to be required.

To perform that search, I designed a second Python program to find maximum partial lifetimes based on user-defined mixing values as well as the raw Yukawa couplings fixed by the fermion sector fitting. Parameter values are defined on a per-trial basis for any number of trials. I started with the most optimistic case by generating random initial values for  $x_i$  and  $y_i$  (but  $x_0 \sim 1$  fixed), with the decay width for  $K^+ \bar{\nu}$  minimized by adjusting those values in each trial. The minimization was again performed using the `Minuit` tool library.

The search based on fully random initial values was unsuccessful, in that the  $K^+ \bar{\nu}$  mode lifetime consistently fell in the  $10^{31-32}$  year-range for the type-II solution and was typically  $\sim 1 \times 10^{33}$  years for the type-I case;<sup>1</sup> at the same time however all five other

<sup>1</sup>The `Minuit` tool used, `Migrad`, works using a local gradient-based algorithm, so that in large

decay mode	baseline for $\tau$ (yrs)	baseline in ref. [19] (yrs)
$p \rightarrow K^+\bar{\nu}$	$8.29 \times 10^{31}$	$6.38 \times 10^{28}$
$p \rightarrow K^0 e^+$	$9.73 \times 10^{34}$	$2.52 \times 10^{30}$
$p \rightarrow K^0 \mu^+$	$5.68 \times 10^{33}$	$6.15 \times 10^{29}$
$p \rightarrow \pi^+\bar{\nu}$	$4.25 \times 10^{33}$	$4.45 \times 10^{29}$
$p \rightarrow \pi^0 e^+$	$1.08 \times 10^{36}$	$3.90 \times 10^{30}$
$p \rightarrow \pi^0 \mu^+$	$6.45 \times 10^{34}$	$6.00 \times 10^{29}$

Table 6.6: Hypothetical baseline partial lifetimes determined using type-II solution Yukawas and  $x_0 = 0.95$  with all other  $x_i, y_i = 0$ . For comparison, I give the analogous results for calculation using type-II Yukawas from the 2010 paper by Altarelli and Blankenburg [19], which use general Yukawa texture. Note in comparing with Table 6.5 that for our model, only the  $K^+\bar{\nu}$  mode fails to satisfy the lower limit, while all modes are well below the limits for the model in [19].

modes in question were usually near or above their respective limits for those same arbitrary mixing values. Hence it was clear even with the  $K^+\bar{\nu}$  mode failure that the ansatz was having the desired effect to some extent. Also, note that this type-I solution for  $K^+\bar{\nu}$  was short of the limit by only about a factor of five. This is surprising since the type-I-based Yukawas reported in eq. (6.5) fell short of meeting the ansatz criteria. Given the differing behaviors of the two solutions, I will report the remaining details in separate subsections once again.

### 6.2.1 Proton Partial Lifetimes for Type II Seesaw

To further explore the properties of the “default behavior” of the lifetime values in the model, I considered the case in which  $x_0 \sim 1$  and all other  $x_i$  and  $y_i$  are set to zero; one can see this case as defining a baseline for the partial lifetimes, in that any  $x_0$  terms in the  $C_{ijkl}$  not suppressed by the Yukawa textures are necessarily large, and whereas problematic contributions from some other  $x_k$  with  $k \neq 0$  may be suppressed simply by setting  $x_k \ll 1$ , the  $x_0$  contributions can be mitigated *only* through cancellation.

The corresponding baseline lifetimes for the dominant modes in the type-II case are given in Table 6.6. One can see that the  $K^+\bar{\nu}$  mode decay width must be lowered by two orders of magnitude through cancellation of  $x_0$  terms by the others. Since it is  $|C|^2$  that appears in the decay width expressions, the needed cancellation amounts to an  $\mathcal{O}(10^{-1})$  tuning among the  $C^A$  factors. Furthermore, as it would be equally

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parameter spaces, initial values are crucial in locating global minima.

unnatural to see  $x_k \ll 1$  for all  $k \neq 0$ , one should expect  $\mathcal{O}(1)$  cancellations to be present anyway; therefore, the needed “tuning” is little more than a very ordinary restriction of parameter space.

In order to elucidate the significance of the improvement created by the Yukawa ansatz, consider the outcome of this baseline calculation for a case with more general Yukawa texture. The model from a 2010 paper by G. Altarelli and G. Blankenburg [19] has the same **10-126-120** Yukawa structure but with general  $h$  and  $g$  as in eq. (4.37) and a tri-bimaximal  $f$  having no hierarchical texture.<sup>2</sup> Using the parameters reported to give a successful fermion fit in the work (*see footnote*), I obtain the baseline results shown in the final column of Table 6.6. One can see here that lifetimes for all modes are far below the experimental limits, by factors of  $\mathcal{O}(10^{3-5})$ ; hence cancellation among the  $C^A$  factors must be  $\mathcal{O}(10^{-2-4})$ . Such sensitive relationships among these factors are seemingly less natural than the result from our model in the absence of some new symmetry.

In order to locate an area of mixing parameter space which yields a sufficient  $K^+\bar{\nu}$  lifetime, I wrote a supplementary Mathematica code to search for minima among strongly abridged versions of  $|C_{\widetilde{W}}^I + C_{\widetilde{W}}^IV|$  and  $|C_{h^\pm}^IV|$  that contribute to the decay width.<sup>3</sup> Specifically I started with  $x_0$  terms only, corresponding to the baseline case, and then iteratively added back the largest contributions one by one while readjusting the initial values each time. Once all of the most important terms were present, I took the resulting mixing parameters as my initial values in the Python code. The resulting minimization gave a large percentage of trials with all six modes exceeding the lifetime bounds.

With an allowed region of parameter space found, I expanded my searches to include a slightly wider range of values for the heavily restricted  $x_0$ . Using six different “seeds” for parameter values, all of which give every mode sufficient with  $\tau(K^+\bar{\nu})$  roughly twice the experimental bound, I created a large number of trials for which the initial values were distributed normally around the seed values and with large standard deviations. The resulting data for such a search is shown in scatter plots below. Figure 6.1 gives the relationships between the  $K^+\bar{\nu}$  mode and other representative modes and also the distribution of  $K^+\bar{\nu}$  lifetime for varying  $x_0$ . Figure 6.2 shows the relationships between other more closely correlated modes for completeness.

Note the strong correlation between  $\pi^+\bar{\nu}$  and  $\pi^0\mu^+$ , which are related by isospin,

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<sup>2</sup>This specific model has already been ruled out due to  $\theta_{13} \sim 6-7^\circ$  typical of tri-bimaximal models.

<sup>3</sup> $C_{h^\pm}^{III}$  and  $C_{h^0}^{III}$  cancel identically for all contributing channels of both the  $K^+\bar{\nu}$  and  $\pi^+\bar{\nu}$  modes.



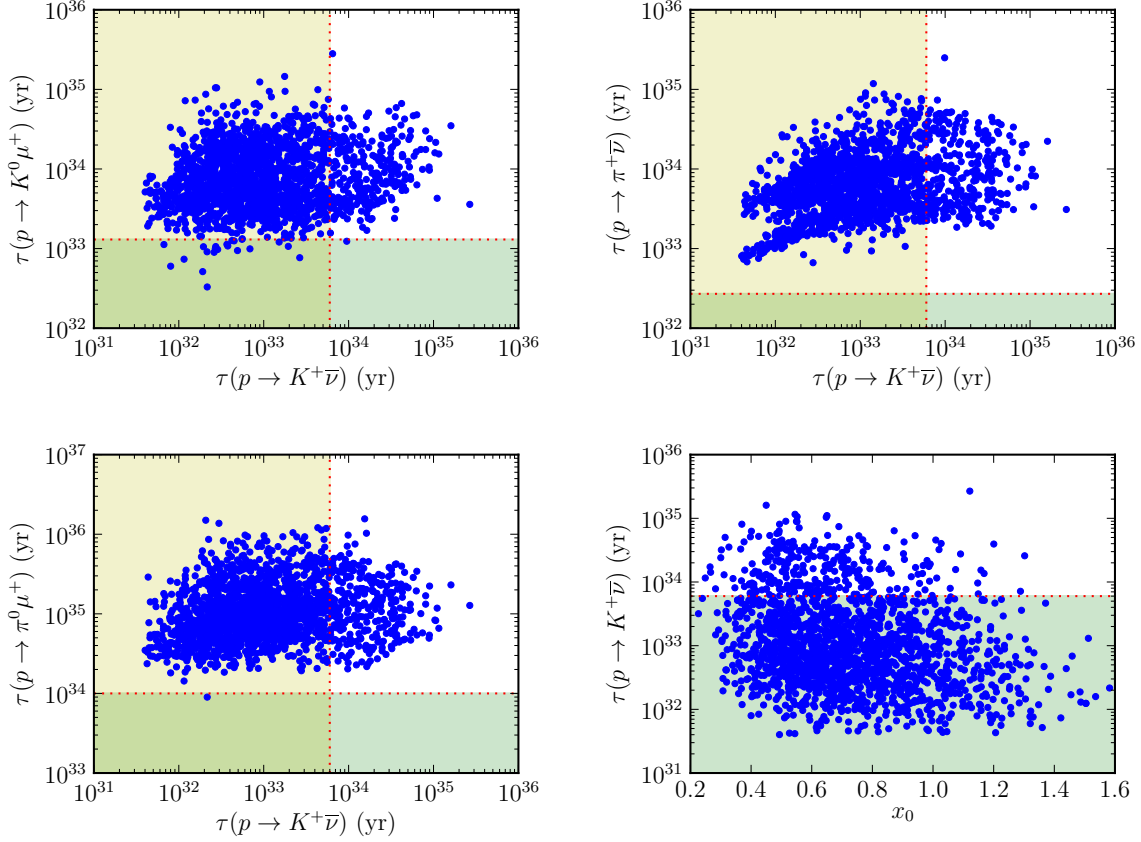


Figure 6.1: Comparisons of  $K^+ \bar{\nu}$  partial lifetime to those of other dominant modes in the model, and that lifetime as a function of the **10** mass parameter  $x_0$ , for the type-II case. Note the unsurprising preference for smaller  $x_0$ .

and the extreme correlation between  $K^0 e^+$  and  $K^0 \mu^+$ . The latter is due to a manifestation of the hierarchical nature of the Yukawas in the  $C_{ijkl}$ , as well as minor features such  $f_{11} \sim f_{12}$ ; similar structure is present in the  $y^f$  and  $U^f$ , which tend to also have  $11 \sim 12$  or  $11 \ll 12$ ; these properties result in a straightforward scaling under the replacement  $l : 1 \rightarrow 2$ . Furthermore, the same relationship is present between  $\pi^0 e^+$  and  $\pi^0 \mu^+$ . These relationships imply that the remaining plots I omitted differ only trivially from the representatives present.

I also performed simple scans in search of a maximum value for  $\tau(K^+ \bar{\nu})$ , as well as taking note of any especially large values in the previous searches. While there does not seem to be any analytically-enforced maximum present in the model, I did consistently find that  $\tau > 10^{35}$  years was extremely rare, and I never saw a value higher than  $\sim 6 \times 10^{35}$  yr. Given those findings, combined with the apparent smallness of

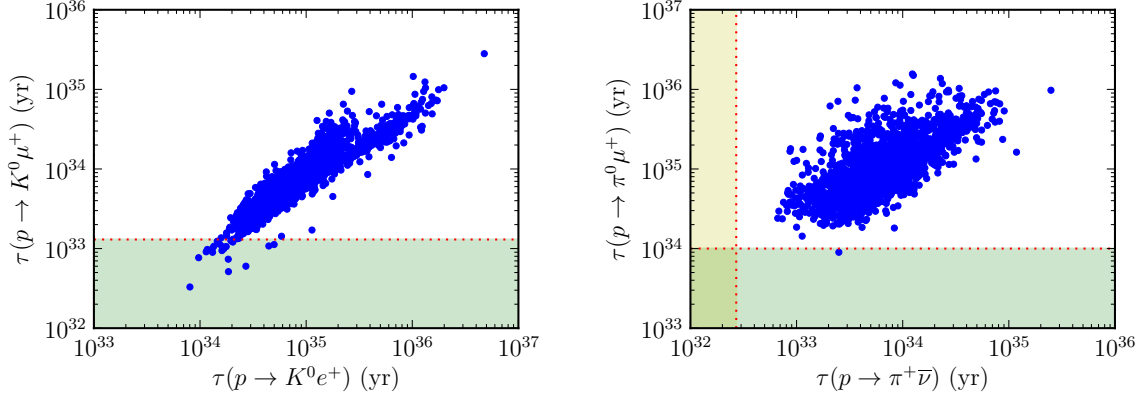


Figure 6.2: Comparisons of partial lifetimes among highly-correlated sub-dominant modes in the model for the type-II case.

the swath of parameter space yielding the above results and the low likelihood of a more global minimum based on my search methods, I believe that  $\tau(K^+\bar{\nu}) \gtrsim 10^{36}$  yr is statistically infeasible in this model for type-II seesaw. If such a value does exist, it is likely contained in a vanishingly small area of allowed parameter space and accomplished through truly extreme tuning. Therefore I will take  $10^{36}$  years as a *de facto* upper limit on  $\tau(K^+\bar{\nu})$  for the type-II case, which will not be accessible by Hyper-K and similar experiments [20, 21] in the near future, but should nonetheless allow the model to be tested eventually.

The other modes of course have similar limits, but it would seem that all the others are substantially higher and thus either far beyond the reach of the forthcoming experiments or beyond the contributions from gauge boson exchange, if not both, with the possible exception of  $\tau(\pi^+\bar{\nu})$ , which is rather highly correlated with  $K^+\bar{\nu}$  in this model. Determining that value is tricky though because if I simply maximize the  $\pi^+\bar{\nu}$  mode, then the  $K^+\bar{\nu}$  mode will be below its bound; thus, there is some question as to how one defines the maximization.

## 6.2.2 Proton Partial Lifetimes for Type I Seesaw

I begin again by examining the same baseline case for the partial lifetimes, with  $x_0 \sim 1$  and all other  $x_i, y_i = 0$ . The resulting values for the dominant modes in the type-I case are given in Table 6.7. Here I find a much more favorable situation, in that even the  $K^+\bar{\nu}$  mode decay width is sufficient, and in fact the other modes exceed the bounds by 2-4 orders of magnitude. Hence I expect that virtually all solutions will be adequate

decay mode	baseline for $\tau$ (yrs)
$p \rightarrow K^+ \bar{\nu}$	$7.87 \times 10^{33}$
$p \rightarrow K^0 e^+$	$5.93 \times 10^{35}$
$p \rightarrow K^0 \mu^+$	$2.45 \times 10^{35}$
$p \rightarrow \pi^+ \bar{\nu}$	$2.37 \times 10^{36}$
$p \rightarrow \pi^0 e^+$	$6.11 \times 10^{38}$
$p \rightarrow \pi^0 \mu^+$	$2.27 \times 10^{38}$

Table 6.7: Hypothetical baseline partial lifetimes determined using type-I solution Yukawas and  $x_0 = 0.95$  with all other  $x_i, y_i = 0$ . Note in comparing with Table 6.5 that all modes satisfy the lower limits, and most do so by several orders of magnitude.

for modes other than  $K^+ \bar{\nu}$ , and as long as there is no *enhancement* due to (de)tuning among the  $C^A$  factors, that mode will be adequate as well.

This is of course a remarkable improvement over traditional models, yet it seems to contradict our expectations given the properties of the fermion fit. Why then is the model successful? There are two primary reasons, both of which are quite subtle. The first reason is that the smaller values for  $g_{13}$  and  $g_{23}$  seen in eq. (6.5) do in fact improve the situation, as I suggested, while the larger  $f_{12}$  and  $g_{12}$  seem to have less impact. Since  $M(h_{33})$  is such an extremely dominant factor in the Yukawas, it is generally the case that contributions involving third generation are larger and more important than the others.

The second reason is even more unexpected, to the point that it was not even examined in the preceding works on this ansatz. The unitary matrices  $U^f$  for the charged fermions are generally  $\sim 1$ , just as one would expect, given the texture of CKM and the absence of any known mixing among charge leptons. This model is no exception, with off-diagonal terms generally  $\mathcal{O}(10^{-1-3})$ ; however, with such sparse or hierarchical (flavor basis) Yukawas due to the ansatz, these “small” off-diagonal elements lead to “small” rotations of  $h, f, g$  resulting in relatively substantial changes to the textures of  $\hat{h}, \hat{f}, \hat{g}$ . Especially noteworthy are the changes in  $h \rightarrow \hat{h}$ , where some previously-zero off-diagonal elements are replaced by the same  $\mathcal{O}(10^{-1-3})$  values seen in the  $U^f$ .

In light of the surprising non-triviality of the basis rotations, if one compares  $U^{u,d}$

for the type-I case:

$$\begin{aligned}
U^u &= \begin{pmatrix} 0.994 & -0.1085 + 0.0057i & 0.00298 + 10^{-5}i \\ 0.1084 + 0.0057i & 0.994 & 0.0047 + 10^{-5}i \\ -0.0035 - 10^{-5}i & -0.0044 + 10^{-5}i & 0.99998 \end{pmatrix} \\
U^d &= \begin{pmatrix} 0.967 & -0.1087 + 0.2309i & 0.00175 + 0.001175i \\ 0.1086 + 0.2308i & 0.966 & 0.03935 + 0.00690i \\ -0.0076 - 0.0072i & -0.0381 + 0.00613i & 0.9992 \end{pmatrix}, \quad (6.6)
\end{aligned}$$

to those for the type-II case:

$$\begin{aligned}
U^u &= \begin{pmatrix} 0.972 & 0.2098 - 0.1044i & -10^{-5} - 0.010i \\ -0.210 - 0.1043i & 0.971 & -0.00012 - 0.0414i \\ -0.0043 - 0.001i & -0.001 - 0.0423i & 0.999 \end{pmatrix} \\
U^d &= \begin{pmatrix} 0.9998 & 0.00633 - 0.0095i & 0.00765 - 0.01117i \\ -0.00708 - 0.0095i & 0.9983 & 0.03386 - 0.04514i \\ -0.00785 - 0.01054i & -0.03401 - 0.04514i & 0.9983 \end{pmatrix}, \quad (6.7)
\end{aligned}$$

one sees that the off-diagonal entries are the same size or smaller for the type-I case in every entry except  $U_{12}^d, U_{21}^d$ ; furthermore, several of the elements involving the third generation are smaller by an order of magnitude. These differences may seem rather benign, but in fact each of these slightly suppressed values individually translates into a factor of 10 suppression in most of the dominant  $C_{ijkl}$ , which all tend to involve third generation elements. In some cases, two or even three such suppressions may affect a single  $C^A$  factor. The squaring of factors in the decay width then gives suppressions of generally 2-4 orders of magnitude in the lifetimes, which is precisely what one can see when comparing Tables 6.6 and 6.7.

Due to the more favorable circumstances, I was able to locate an allowed region of parameter space for type-I simply by running a large number of trials with the type-II parameter seeds. I repeated the process of expanding the range of  $x_0$  by again choosing five seeds that gave every mode as sufficient and  $\tau(K^+\bar{\nu})$  roughly twice the experimental bound, and I again used those seeds to create scatter plots for a large number of trials. Figure 6.3 gives the relationships between the  $K^+\bar{\nu}$  mode and other representative modes and the distribution of  $\tau(K^+\bar{\nu})$  as a function of  $x_0$ , and Figure 6.4 shows the relationships between other more closely related modes. Note the bifurcation

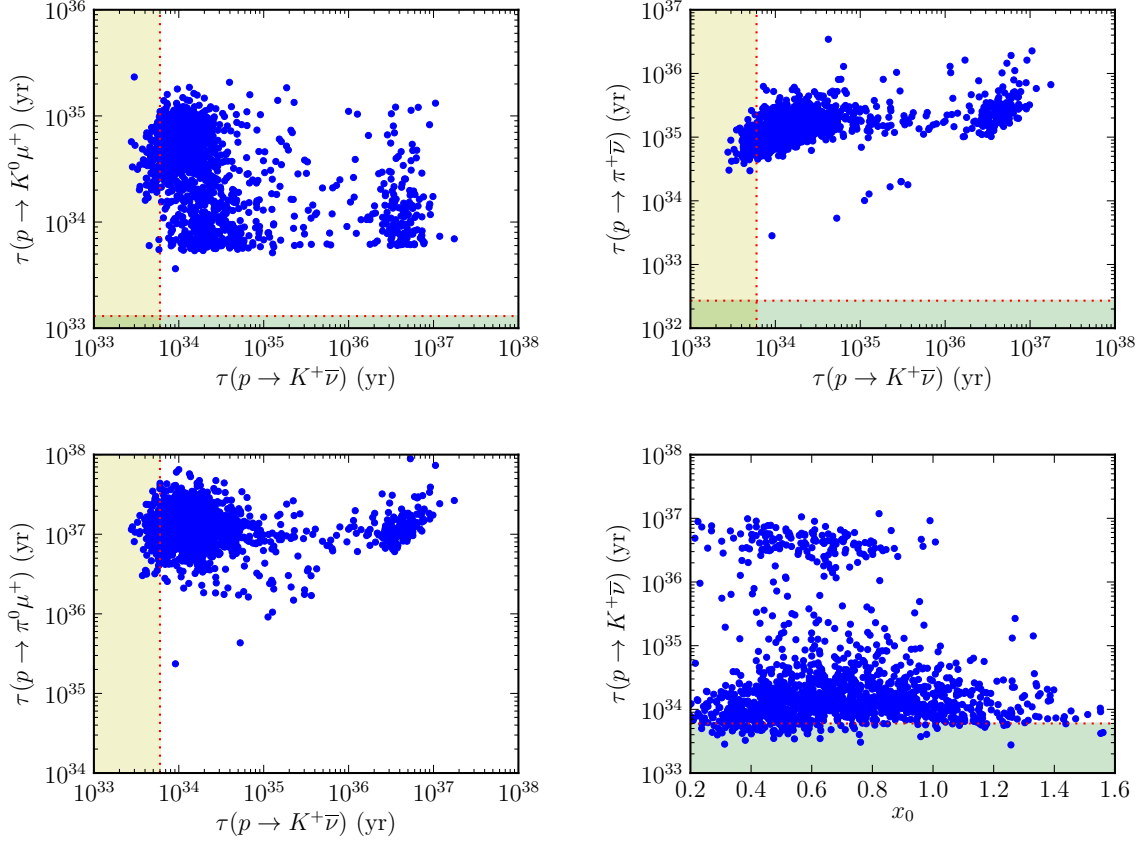


Figure 6.3: Comparisons of  $K^+\bar{\nu}$  partial lifetime to those of other dominant modes in the model, and that lifetime as a function of the **10** mass parameter  $x_0$ , for the type-I case. Note the unsurprising preference for smaller  $x_0$ .

of the solution set in each plot; I have not yet been able to discover the cause of this behavior.

Again I performed scans to determine a statistical upper bound for the value of  $\tau(K^+\bar{\nu})$  in the model. I consistently found that  $\tau > 10^{37}$  years was rare and did not see a value higher than  $\sim 3 \times 10^{37}$  yr. Given those findings, I suspect that the *de facto* upper limit on  $\tau(K^+\bar{\nu})$  for the type-II case is slightly lower than  $10^{38}$  years for the type-I seesaw case. Such a value is certainly out of reach of Hyper-K and other imminent experiments. Note that as values for the neutral Kaon and pion lifetimes often exceeded  $10^{38}$  years in my findings involving  $K^+\bar{\nu}$  minimization, the upper limits for those modes are surely sub-dominant to gauge exchange as well as out of reach of experiments and so not of interest.

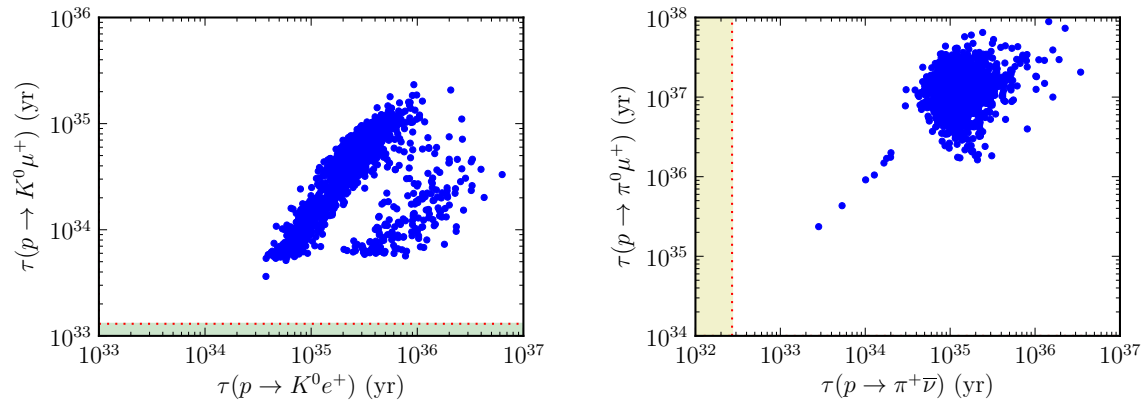


Figure 6.4: Comparisons of partial lifetimes among highly-correlated sub-dominant modes in the model for the type-I case.

# Chapter 7

## Conclusion

In this work I have presented a full analysis of the nature of proton decay in an  $SO(10)$  model that has **10**,  $\overline{\mathbf{126}}$ , and **120** Yukawa couplings with restricted textures intended to naturally give favorable results for proton lifetime as well as a realistic fermion sector. The model is capable of supporting either type-I or type-II dominance in the neutrino mass matrix, and I have analyzed both types throughout.

Using, numerical minimization of chi-squares, I was able to obtain successful fits for all fermion sector parameters, including the  $\theta_{13}$  reactor mixing angle, and for both seesaw types. Using the Yukawa couplings fixed by those fermion sector fits as input, I then searched the parameter space of the heavy triplet Higgs sector mixing for areas yielding adequate partial lifetimes, again using numerical minimization to optimize results. For the case with type-II seesaw, I found that lifetime limits for five of the six decay modes of interest are satisfied for nearly arbitrary values of the triplet mixing parameters, with an especially mild  $\mathcal{O}(10^{-1})$  cancellation required in order to satisfy the limit for the  $K^+\bar{\nu}$  mode. Additionally, I deduced that partial lifetime values of  $\tau(K^+\bar{\nu}) \gtrsim 10^{36}$  years are vanishingly unlikely in the model, implying the value can be taken as a *de facto* lifetime for the mode, which makes the model ultimately testable. For the case with type-I seesaw, I found that limits for *all six* decay modes of interest are satisfied for values of the triplet mixing parameters that do not result in substantial enhancement, with limits for modes other than  $K^+\bar{\nu}$  satisfied for nearly arbitrary parameter values; furthermore, I deduced a statistical maximum lifetime for  $K^+\bar{\nu}$  of just under  $10^{38}$  years.

Given these results, I conclude that the well-motivated Yukawa texture ansatz proposed by Dutta, Mimura, and Mohapatra is a remarkable phenomenological suc-

cess, capable of suppressing proton decay without the usual need for cancellation, and without compromising any aspect of the corresponding fermion mass spectrum. This result stands out among similar analyses and perhaps represents a generally more favorable approach for understanding the suppression of proton decay in grand unified theory models.



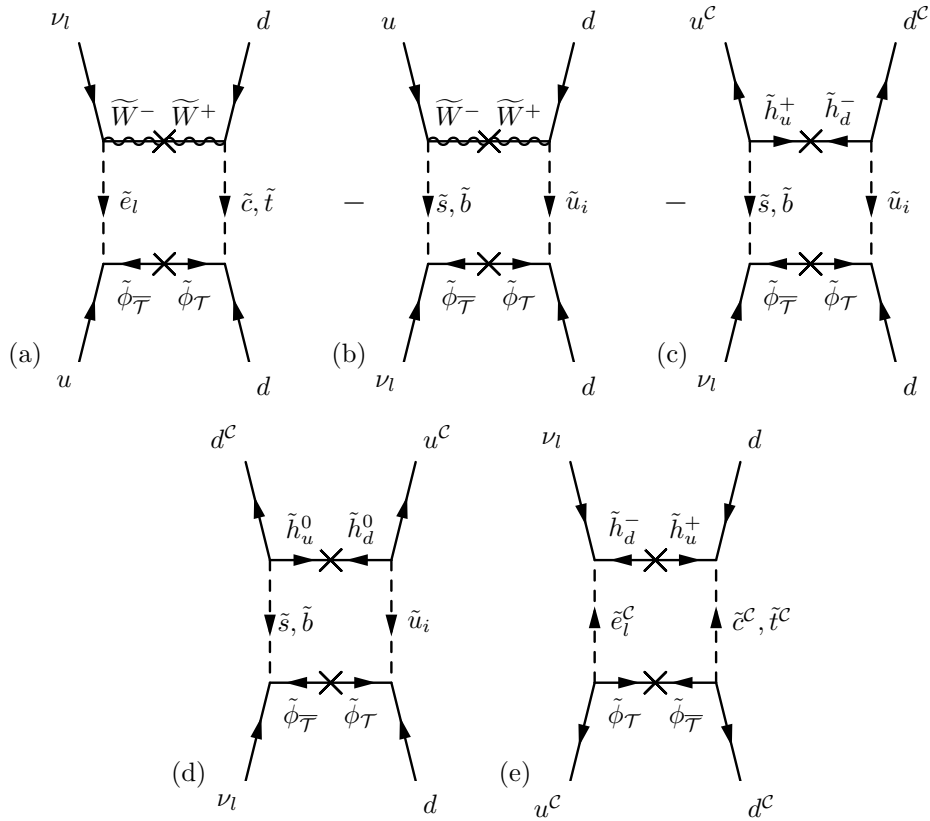
# Appendix A

## Feynman Diagrams for Dimension-6 Operators Contributing to Proton Decay

$\tilde{\phi}_{\mathcal{T}}$  is the Higgsino component of a heavy color-triplet Higgs superfield;  $\phi = H, \bar{\Delta}, \Sigma$ .

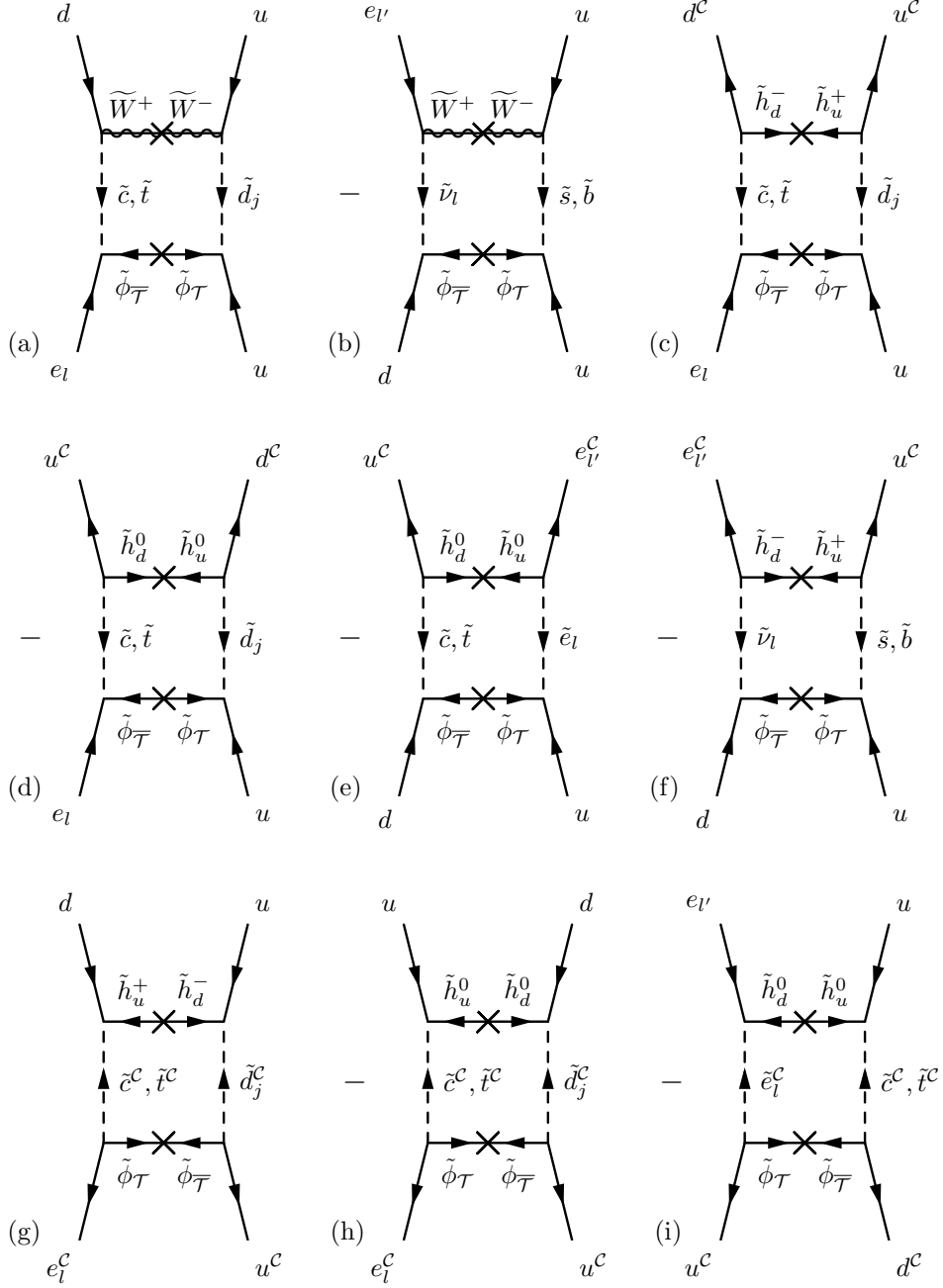
**Channels for  $p \rightarrow \pi^+ \bar{\nu}$**

$i, l = 1, 2, 3$ .



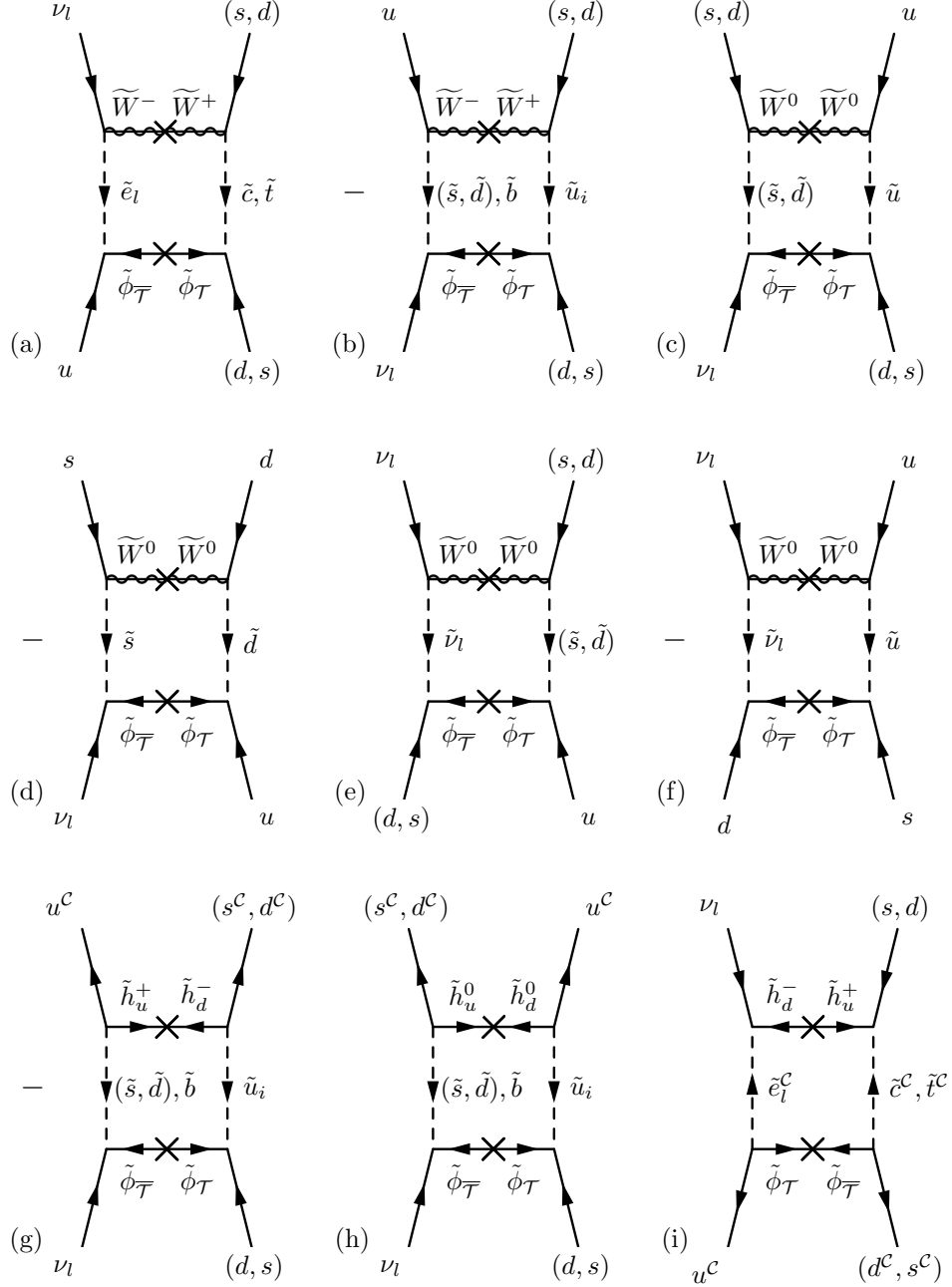
**Channels for  $p \rightarrow \pi^0 \ell^+$**

$j = 1, 2, 3$ ;  $l = 1, 2$  ( $\leftrightarrow \ell = e, \mu$ ), or for diagrams including  $l'$ , instead  $l = 1, 2, 3$  and  $l' = 1, 2$ .



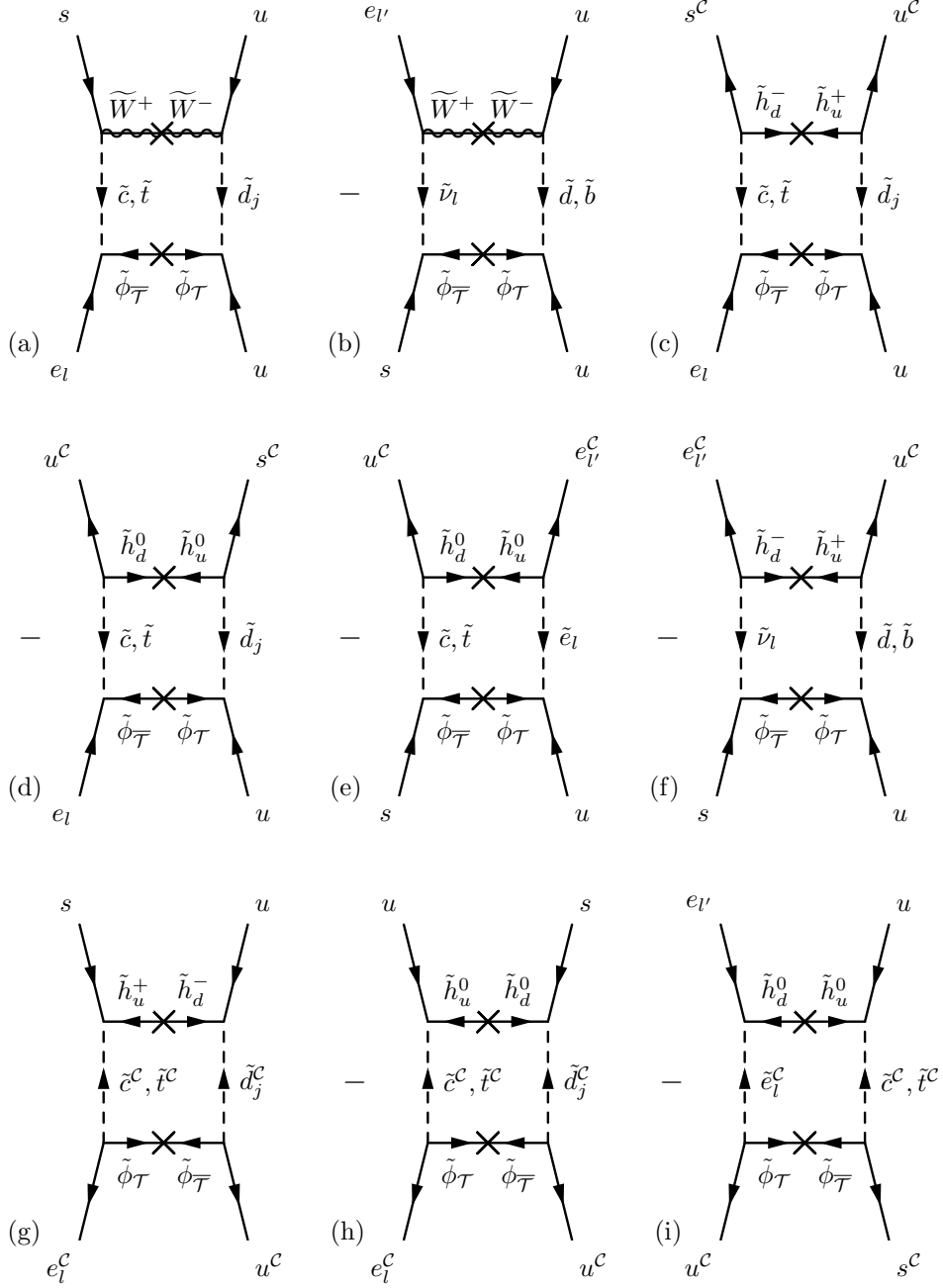
### Channels for $p \rightarrow K^+ \bar{\nu}$

$i, l = 1, 2, 3$ ; parentheses indicate coupled choices; absence of diagrams for  $\tilde{u}du\tilde{e}$  dressed by  $\tilde{h}^\pm$  and  $u\tilde{d}\tilde{d}\bar{\nu}$  dressed by  $\tilde{h}^0$  is due to resulting external  $\nu^c$ .



**Channels for  $p \rightarrow K^0 \ell^+$**

$j = 1, 2, 3$ ;  $l = 1, 2$  ( $\leftrightarrow \ell = e, \mu$ ), or for diagrams including  $l'$ , instead  $l = 1, 2, 3$  and  $l' = 1, 2$ .



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